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OXON

A ROYAL ROAD
TO
GEOMETRY;
OR, AN
EASY AND FAMILIAR INTRODUCTION
TO THE
MATHEMATICS.
IN TWO PARTS.

I. PRACTICAL GEOMETRY, with Applications, and a familiar Introduction; for the use of Mechanics, &c. Alſo, the Conſtruction of the Ellipſis; with ſome of its chief Properties demonſtrated.

II. ELEMENTS OF GEOMETRY ABRIDGED, Containing the whole Subſtance of Euclid's firſt ſix, the eleventh and twelfth Books; with ſeveral other, uſeful and valuable, Theorems; treated in the moſt brief, eaſy, and intelligent manner; for the uſe of Schools, &c.

Being an Attempt to render that moſt uſeful and neceſſary Science more univerſal, and practically applicable.

With Notes interſperſed, critical, explanatory, and inſtructive.

By THOMAS MALTON.

To which is annexed, an Appendix, on the Theory of Menſuration of Superficies and Solids, as deduced from the Elements.

LONDON:
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And ſold by Meſſ. ROBSON, Bond-Street; BECKET, Strand; ROBINSON, Pater-noſter Row; TAYLOR, in Holborn; and by the AUTHOR, in Poland-Street, Oxford-Road.
MDCCLXXIV.

TO

SIR CHARLES FREDERICK,

Knight of the most honourable Order of the Bath, F. R. S. &c.

SIR,

AS there is an undisputable Propriety in dedicating to a competent Judge of a Science, an Attempt to familiarize the Elements of it, I hope that propriety will plead an excuse for this Presumption; especially, as the Subject is so nearly connected with the business of Your Personal Station, and too highly important to the Publick, in general, to conceive, that any attempt at a clearer Investigation can be beneath the notice of the Surveyor-General of his Majesty's Ordnance, and one of the Representatives of the People.

How far my Method of treating the Elements of Geometry may merit your Approbation, I presume not to imagine; farther than, as I flatter myself, the simplicity with which I have endeavoured to elucidate

the firſt Principles, to render the knowledge and the uſe of Geometry more general, and practically uſeful in common Life, where it is moſt wanted, may prove an additional inducement to your excuſing the liberty I have taken, in thus ſoliciting Your Patronage and Protection; for a Work, whoſe Subject, alone, is ſufficient to juſtify the attention You may pleaſe to honour it with; whatever may be thought of the manner in which I have treated it; having been, frequently (in order to compriſe the whole Elements in leſs compaſs) under the neceſſity of deviating, conſiderably, from the Path of Euclid.

I am,

SIR,

with due reſpect,

Your moſt obedient

humble Servant,

THOMAS MALTON.

ADVERTISEMENT.

IT may appear ſomewhat ſtrange to call ſo copious a Work as this, an Abridgment; which, on examination, and comparing with thoſe who have tranſcribed Euclid, will be found to be a great abridgment of the Elements, both in the number of Theorems and length of the Demonſtrations.

But, they will find, in this Work, particularly in the 2nd, the 3rd, the 6th, 7th and 8th Books, ſeveral valuable Theorems which are not Euclid's; ſome of which are elementary, and really neceſſary to be known. Upon the whole, there are upwards of eighty other Theorems and Problems, excluſive of the Ellipſis.

In reſpect of the practical Part, with the Introduction, it may be deemed a compleat Work of itſelf; which, with the Appendix, is more than one third part of the whole.

The Applications, Notes, and Remarks, neceſſary to a young Student, and the length of ſome (in the Fifth Book particularly) have ſwelled the Work conſiderably. The much greater number of Terms defined; various ways of performing the ſame Problem, and alſo of demonſtrating ſome Theorems; the number of Figures, and the Preamble to each Book, &c. alſo the manner of printing, in order to have the Figure always in View, ſo that, many Pages are not near full; all which have concurred to extend the Work, greatly beyond my firſt Deſign; yet, when compared with the cloſe printing of others, the Elements will be found abridged, almoſt half; nevertheleſs, it contains the whole, in Subſtance. The full and perfect knowledge of all, and more than is contained in Euclid, may be acquired in a third part of the Time, and with infinitely more Eaſe, Pleaſure, and Satisfaction to the Learner (having no *Aſſes Bridge* * to get over, but the Road ſmooth and even) conſequently, it may, with great Propriety, be called an Abridgment.

* *The 7th (ſome ſay the 5th) of the firſt Book, hat, in the Seminaries of Erudition, been long known by the Appellation of the* Aſſes Bridge; *on which, many have foundered and never got over; nor been able to advance one ſtep further.*

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N. B. The Problems, of Euclid, are collected together, in the firſt Part; amongſt other, ſelect Problems, in Practical Geometry.

Note. The Numbers in the Firſt Column are Euclid's; and, oppoſite to them is ſhewn where each Propoſition may be found in this Work, whether Problem or Theorem.

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PREFACE.

THE mathematical World will, I doubt not, be furprized at a frefh publication of the Elements of Geometry, by one entirely unknown ; and, on a plan very different from that of others who have wrote on the Subject. I hope they will fufpend their opinion, and not pafs a too hafty cenfure, on account of the obfcurity of the Author, till after they have given it a fair and candid perufal, and then proceed to judgment with candour and impartiality.

I do not pretend to much knowledge in the Mathematics, having been brought up in a way of life, very different from my inclination ; yet, what time I could fpare from bufinefs and the demands of my family, I chofe to employ in fuch ftudies ; and have, by dint of ftudy, only, and without any other inftruction, made fome progrefs in mathematical Sciences ; of which, Geometry is the firft, and a fure key to the reft.

Since I have made myfelf, by felf-application, a Proficient in Geometry, and have made fome branches of the Mathematics my Study and Profeffion, I have often been furprized at the negligence and deficiency of our common Schools, for the cultivation of Youth who are intended to fill the middle fphere of Life, in mechanic Trades, &c. They, almoft in general, purfue one common Plan or track of Learning. After the firft and neceffary branches, Reading, Writing, and Arithmetic ; which, indeed, might be acquired in half the time it ufually is ; the next ftep (if the Pupil has made a progrefs thro' Arithmetic in any reafonable time) is the Grammar of the Latin Tongue through which, he fweats and labours to little purpofe. If the Pupil has three or four years to fpare, before he goes out to bufinefs, he perhaps gets into the Cordery or Erafmus ; or, if he reaches Cornelius Nepos, he is looked on as a prodigy.

Now, it may reafonably be afked, for what purpofe all this Time has been fpent ? which might have been employed to much

better purpofe. For, what has mechanic Trades to do with Latin? any more than a common Porter or Carman with Logic; it may indeed complete him a Pedant or Coxcomb, but can never be of real ufe in his Profeffion; even fuppofe he had made a tolerable proficiency, it could anfwer no purpofe but to fet him above his Employment, without being of any fervice in it.

On the contrary (fuppofing no particular avocation is intended for the Student) if, inftead of Latin, Geometry and Menfuration, &c. were introduced in all public, common Schools, I would afk any perfon, who has confidered thefe things, and their ufes in Life, which is moft likely to turn to the Pupil's advantage? Is there a mechanic Profeffion in which Geometry or Menfuration may not be of fome ufe? in fome particular ones it is well known to be of the greateft, the foundation of it; and yet, altho' the Youth was particularly intended for that Profeffion, it was, perhaps, never once fo much as thought on; until, by too late experience, he finds the want of it: I mean all fuch Trades as relate particularly to Building, in general. Had fome Builders, whom I have known, been converfant in the Mathematics, or only in plane Geometry; inftead of plodding on in a low fphere of Employment, they would, if their natural, mechanical genius had been properly cultivated, have filled a more elevated ftation.

For the ufe of fuch, I have been at the trouble of compofing this Treatife. If only the practical part is well inculcated, it will be of more fervice, in common Life, than a proficiency in Latin can poffibly be. If the young Pupil has a genius, and difcovers a relifh for mathematical Science, let him go on with the Elements; and if he acquires a competent fhare of knowledge therein, it will then be time to confider, what particular Profeffion he either wifhes or is deftined for. In choofing of which, regard ought particularly to be had to the Boys genius and difpofition, which will, ere this, be difcernable. But, at all adventures, inftead of flogging and driving a ufelefs dead Language into a ftupid Boy, which only renders him more fo, let the practice of Geometry be introduced in its ftead, in every common School; there is fomething entertaining

to

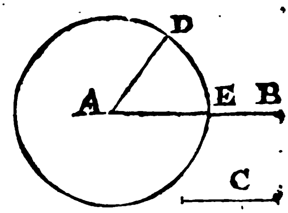
to the Mind, more than in burdening the memory with *As in præsenti*, and other rules of the Latin Grammar.

Accuſtoming Boys, early, to handle the Compaſſes and other drawing utenſils, in delineating all the Diagrams, as they proceed, will be an entertainment to them, and of great utility, rather than a perplexing ſtudy, and gradually enure them to Demonſtration ; which, under a proper Tutor, they would ſoon have a reliſh for, and then they would proceed with pleaſure : beſides, it is an introduction to Drawing. A foundation being once laid in Geometry, they are then qualified to purſue any other branch of the Mathematics, ſuitable to the Profeſſion they are intended for ; ſuch as Menſuration, Trigonometry, Navigation, Gunnery, Fortification, Architecture, naval or domeſtic, Surveying, &c. In ſhort, all the uſeful and neceſſary Employments, in the mechanic Arts, have their foundation in this moſt neceſſary Science ; which, being acquired, will, moſt probably, make its Poſſeſſor ſtrike out of the common and vulgar track, and make him eminent and diſtinguiſhable, in whatever Profeſſion he is caſually fixed in ; as he will have laid a ſolid and permanent foundation, in Theory ; whereon, may, very probably, be erected, a laſting monument to his future Fame.

I have peruſed ſeveral Authors on the Subject, and find, that ſome have treated it in a manner ſcarce intelligible to a beginner, unleſs he has ſome knowledge of Algebra ; others would be better underſtood and approved, if they did not dwell too much on ſelf-evident Propoſitions, which are, in themſelves, perfect Axioms. Perhaps, I ſhall be blamed for cenſuring, as uſeleſs, ſeveral Propoſitions in that famous Geometrician, Euclid ; but muſt own, I cannot conceive of what uſe is all that tedious round about method, in the 2nd Problem, Book 1. *viz.* "To "put a right Line, at a given point, equal to a given Line," unleſs ſome particular direction, in reſpect of the other Line, was alſo given.

In Problem 3d. where the 2nd is applied, I aſk, for what uſe? and why, after having taken the line C in the Compaſſes, as Radius,

we may not as well cut off from the given Line (AB) at once, the meaſure of C, (at E) without firſt placing it from A to D? (See the Figure) nor can I perceive that the Demonſtration is, at all, the clearer for it.



As the knowledge I have acquired, in ſuch ſtudies, has been entirely from Books, without any other aſſiſtance; I may, perhaps, have been able to ſee deficiences or ſuperfluities in them, better than thoſe who have ſtudied under a Tutor; and, I dare venture to affirm, that very few, who are not already tolerably verſed in Geometry, will be able to form any Idea of what the 7th Propoſition, Book I. attempts to prove; who, from inſpection only of a proper Figure, might be fully convinced of the truth of it.

I have, in this Treatiſe, endeavoured to render every Propoſition eaſy and intelligible, to any capacity; and have omitted none that are uſeful, or neceſſary to demonſtrate the reſt; and, I will be bold to affirm, that thoſe who cannot, from it, acquire a knowledge of all that is requiſite, in Plane Geometry, will never be able to attain it at all. Where there is not a capacity to underſtand, they had better deſiſt from the undertaking. An equal talent is not given to all; and, for ſuch as have not a talent and a natural propenſity, to expect to attain any tolerable ſhare of knowledge, in Geometry, is aiming at impoſſibilities.

The 16th and 17th Propoſitions, Book 1ſt, are entirely uſeleſs; for ſince, in the 32nd, the external Angle is proved to be equal to the two remote Angles of the Triangle; and, that all the three Angles, of every Triangle, are equal to two right ones; it ſeems abſurd to prove, before hand, that any two of the Angles are leſs than two right Angles, and, that the external Angle, is greater than either of the remote ones. As if one ſhould undertake, firſt, to prove that five is greater than either two, or three, and afterwards, that it is equal to them both. I have, therefore, made free, to alter, in ſome meaſure, the Elements of the firſt Book; that, by a different arrangement, in tranſpoſing the places

of

of ſome Propoſitions, others, which depend on them, may with more eaſe and elegance be demonſtrated.

I ſhall ever be of opinion with Tacquet, and ſome others, that, to attempt a formal Demonſtration of Propoſitions which are ſelf-evident, is involving a thing, in itſelf clear and conſpicuous, in darkneſs and obſcurity. I have always found more difficulty in demonſtrating, to another Perſon, ſelf-evident Propoſitions, than the moſt intricate of others; and, when done, have only confuſed the Idea which the Pupil had more perfectly from inſpection of the Figure. If a knowledge of ſeveral properties of Figures be acquired, which is neceſſary to elucidate other more ſublime Propoſitions, is not the eaſieſt method the moſt eligible? certainly it is; and, barely to be told ſeveral properties of Triangles is ſufficient for attaining the reſt. Therefore, in this Work, I have reduced ſome Propoſitions into Corollaries; as they are but a certain conſequence of the preceding Propoſition, or of ſome other.

Dr. Keill, in his Preface to his Tranſlation of Commandine, and alſo Mr. Cunn ſeems to think it an unpardonable fault in Tacquet, to omit the Demonſtrations of the 5th Book; and aſſerts, that not one Demonſtration, in the 6th, 11th and 12th, can be obtained without it. But, I muſt beg his pardon, for differing from him in opinion, and am, myſelf, a living witneſs againſt ſuch his aſſertion; for, I do aver, that, without any other demonſtration of the 5th Book, than what Tacquet has delivered, I have been able to go through all the other; and, without vanity, I think I underſtand them; but, of that, let this Treatiſe bear witneſs. I never could, at firſt, have had patience to go through the dry and tedious Demonſtrations, delivered in 25 Propoſitions, of the 5th Book of his Euclid; and, had it firſt fallen in my way, I ſhould certainly have lain it aſide before I had got through a fourth part of it; yet, I muſt acknowledge, that Tacquet is as much too brief as the other is tedious.

I cannot think it abſolutely neceſſary, in order to obtain a competent knowledge in Geometry, and of Proportion in particular, to tread exactly in the ſame ſteps with Euclid; therefore, I have

made

made free to deviate, where I think it for the better, a readier or clearer way of inveſtigating. I have, likewiſe, ſeparated the Problems from the Theorems; making, of them, a diſtinct and ſeparate Book, for the uſe of Schools and for Mechanics in general; the full Demonſtration, of which, may be obtained from the Elements, to which I refer where it is neceſſary. I have given it the firſt place, contrary to ſome others who have made it the laſt Book, or otherwiſe diſpoſed of it. By which means, we are frequently at a loſs in the References, and are told to form Conſtructions, before we have learned how.

My reaſon for ſeparating Practical Geometry from the Theorems is, that there are numbers of People, to whom it may be of great uſe, who either have not leiſure or inclination to go through, or, perhaps, a capacity to underſtand the Demonſtrations of the previous Propoſitions, which diſcourages them from proceeding. But, ſuch as are not inclined to take it on credit, will readily obtain the Demonſtration from the Theorems.

To make the Work more compleat I have added, after Practical Geometry, a brief Theory of the nature and conſtruction of an Ellipſis, that moſt uſeful and valuable Figure. And alſo, as an Appendix to the whole, I have given a conciſe Theory of menſuration of Superficies and Solids; ſhewing their immediate and abſolute dependence on Geometry.

I have for ſome time debated, with myſelf, whether I ſhould publiſh a tract of Geometry or not; from which I have been deterred, through the perſuaſions of ſome particular Acquaintance; alledging, that there are more learned Treatiſes already publiſhed, than any I could produce: the truth of which, I readily aſſent to; but, I don't find that they are more intelligible for that, and have, therefore, determined to publiſh. My chief reaſon, for which, is to bring the whole Elements into leſs compaſs, and to abridge it of that tedious prolixity, which is in many Authors, on that Subject; in dwelling too long on ſuch Propoſitions as are clear and evident of themſelves; and, by that means, to render the ſtudy of Geometry more pleaſant and entertaining. 'Tis enough to deter any one from the purſuit of a Science, who find, in the Rudiments of it,

ſo much perplexity; for, I am perſuaded, that, unleſs a Perſon has a ſtrong inclination to it, and a good natural Capacity, 'tis not a very pleaſing Study, at the firſt, until they begin to feel the ſublime Truths it contains; and am therefore of opinion, that, the eaſier it is made in the begining, the more entertaining to young Students.

Can there need Demonſtration, that any two ſides of a Triangle, are greater than the third? is there a Perſon ſo ignorant as not to know it? it is implanted in us by Nature; every common Porter knows it, or practices it every Day. Who ever ſaw one of them traverſe two ſides of a Square, when he could croſs the Diagonal? and why is it? but, becauſe he knows it to be ſhorter than the two Sides. Is it not obvious, that the greateſt Angle of every Triangle, muſt be oppoſite to the greateſt Side? and can there be any need to demonſtrate the converſe? that the greater Side ſubtends, or is oppoſite to the greater Angle; is not the one contained in the other? it is trifling, to no purpoſe; for, all converſe Propoſitions may be Corollaries to the former. If two Sides of a Triangle, equal to two Sides of another Triangle, contain a greater or leſs Angle, the Baſe will be greater or leſs? Are not all theſe, and ſeveral more, obvious and clear, from a bare inſpection of the Figure? nay even without it; 'tis enough to be told they have ſuch properties, and not to loſe time in trying the patience of the Student, with a tedious and puzling Demonſtration of what he ſaw clearer before; for, if the thing is ſeen or known, what needs there more? is it intended to perplex, only, where it can be of no uſe? to diſguſt the beginner, before he is able to ſee any of the Beauties it contains? Yet, I do not omit theſe entirely, becauſe the whole Elements depend on them; but have endeavoured to treat them in as ſimple a manner as the nature of the Subject will admit of; if I have been conciſe, I doubt not I ſhall be excuſed, if I have but ſaid enough.

But, as I think I have ſaid enough, in this place, I ſhall ſtraightway proceed to the Subject; through which, if the Reader be inclined to follow, with an intention to learn what it contains, I am much

much miſtaken if he loſes his labour; and, for ſuch as peruſe it only with intent to cavil, I hope they will be greatly diſappointed, and find but little to cavil at. I am not ſo vain as to ſuppoſe it is without defect, or that it will pleaſe all, for that I know is impoſſible; but, if I have made it intelligible and uſeful to common capacities, it is what I aimed at, and ſhall reſt ſatisfied in the ſuppoſition that my labour is not entirely loſt.

The greateſt fault in Tacquet is, that his Figures are too ſmall and trivial; and it is a general fault, that they are often incorrect and frequently contradict the deſcription. Correct and well adapted Schemes are certainly of ſome conſequence, in which, I have been very particular; and have, alſo, carefully reviſed the letter preſs, ſo that, I hope there are but few errors have eſcaped my obſervation. For, Errors, in miſplacing the Referrences to the Schemes, and ſometimes omitting them entirely (which is better of the two) is unpardonable in mathematical Works; having frequently experienced, in moſt Authors that I have peruſed, the perplexity it occaſions; eſpecially, when the ſubject is quite new to the Student. But, if any ſhould remain unnoticed, I hope the candid reader will impute it to human fallibility, and not quarrel with it on that account, for I am certain he will not meet with many.

Although I have, in this Treatiſe, deviated greatly from Euclid, in many particulars, I have endeavoured to make it generally uſeful; by putting his Numbers after mine, to each Propoſition, and alſo by means of an Index, I have ſhewn where to find any Propoſition of Euclid; which, in caſe of reference, to Euclid, in other Works, may be readily turned to.

I have well conſidered and digeſted every Propoſition, have carefully reviſed them over and over with the ſtricteſt attention, and I am fully convinced, that there is no omiſſion of any thing that is eſſential, or neceſſary to be known. Notwithſtanding, I have abridged the whole Elements, particularly the firſt, the third, the fifth, and the eleventh Books; yet, I dare venture to affirm, that I have not omitted the ſubſtance of any Propoſition which will ever be referred to, by Authors in any other Science.

AN
INTRODUCTION
TO
GEOMETRY,

Containing a full Definition of the Terms peculiar to, or made uſe of in that Science, with explanatory Notes and Remarks, where it is neceſſary to illuſtrate or enlarge.

Likewiſe a ſhort Theory of Plane Angles; in which they are more fully explained than in any other Work of the kind, that I have ſeen. Indeed, moſt Authors in Geometry are entirely deficient in that reſpect; for want of which, the young Geometrician is frequently at a loſs, to conceive a clear Idea of Angles. I have therefore, been explicit on that Head.

It alſo contains an Explanation of all the Abbreviations made uſe of in this Work; many of which are explained in Engliſh Grammars, and other ſchool Books, and ought to be known to every Engliſh Reader; yet, as I know that their true ſignifications are not ſo generally underſtood, it may not be thought impertinent, or ſuperfluous here.

Nor have I omitted any thing, that is neceſſary to elucidate the Subject I am about to treat on; at leaſt, I think I have not; for, in the courſe of my own ſtudy of it, and in teaching others, I have been enabled to diſcover what is needful to moſt Capacities; and if I have any pretence to merit in this Work, it is chiefly on that account; in rendering an intricate, yet generally uſeful and moſt neceſſary Science, attainable to ordinary Capacities; and, at the ſame time, I hope, not exceptionable to thoſe of acuter talents.

I flatter myſelf that it will not be leſs acceptable to any, for being eaſy to be attained, to write only for Proficients, is to little purpoſe. By ſuch I may, in ſome caſes, be thought

rather prolix; yet, I presume, not tedious; a repetition is sometimes necessary to young Minds, and is more agreeable, in general, than turning back, which they are too frequently obliged to do; it being impossible to retain, in memory, all that is passed over on the first perusal.

GEOMETRY, according to its original derivation, signifies to measure the Earth. It is a Science which contemplates continued Quantity, Extension or Magnitude, abstractedly considered; it teaches the nature and properties of Lines, Angles, Figures, Surfaces and Solids.

Geometry is in two parts, speculative and practical; the first demonstrates the properties of Right Lines, Figures, &c. in speculation; from which is deduced the practical part for various uses, for the benefit of mankind, in mechanic Arts, &c.

Euclid has judiciously divided the Subject into Books or Sections; each of which, treats of different Figures, or different properties of Figures, the power of Lines, Proportion, &c. which some Authors have thought proper to deviate from, though without any justifiable reason for so doing.

It treats, in the first, third, fourth, and sixth Books, of Plane Figures, and thence is called Plane Geometry; and afterwards, in the 11th and 12th of Euclid, the 7th and 8th of these Elements, of Planes and Solids.

Each Book contains sundry Propositions; from which, are deduced Corollaries, Scolia, &c. the signification of all which I shall first beg leave to explain or define; and then proceed to the Definitions of the more essential Terms, which are the Subject of Geometry.

A DEFINITION is the defining or explaining the full signification of any Term or particular Word, peculiar to, or made use of, in that Science of which we are about to treat.

A PROPO-

A PROPOSITION is either a Theorem, propoſed to be proved or demonſtrated, contemplatively; or, it propoſes ſomething to be done, problematically or mechanically.

A CONVERSE PROPOSITION is the contrary of the other; that, which in the foregoing was the Concluſion, drawn from the Premiſes of it.

e. g. If a Triangle have two equal Sides, the Angles which they ſubtend are alſo equal; the Converſe is, that if a Triangle have two equal Angles, the Sides ſubtending them are alſo equal.

A THEOREM is a ſpeculative Propoſition; a declaration of certain properties, equality, or other proportion relative to Quantity, or Figure, mathematically conſidered.

A PROBLEM, is a Propoſition which propoſes ſomething to be done, practically or mechanically.

An AXIOM is a ſelf evident Propoſition, which does not require to be demonſtrated. See Axioms, Book 1. El.

A LEMMA is a Propoſition, as it were by the bye, or out of the way, which ſerves, previouſly, to prepare the way for the more eaſily comprehending the Demonſtration of the following Propoſition.

I do not make uſe of Lemmas in this Work, as ſome geometrical Authors do; for if there be a neceſſity for a Lemma, I ſee no reaſon why that Lemma is not as much a Propoſition as any other. The 7th and 16th Propoſitions of the firſt Book of Euclid, may be called Lemmas, for they are certainly redundant Propoſitions. In other mathematical Works, Lemmas are frequently neceſſary, but, in Geometry, they are quite inconſiſtent.

A COROLLARY is a neceſſary conſequence deducible from ſome Propoſition, already demonſtrated.

A SCHOLIUM is a remark, or uſeful leſſon derived from the preceding Propoſition.

A POSTULATE is a petition or requeſt which is required to be granted. See Poſtulates, p. 21.

HYPOTHESIS. Whatever is ſuppoſed or premiſed, in a Propoſition, is called the Hypotheſis or Premiſes of it; from which ſome certain Conſequence is deduced, as affirmed, and afterwards demonſtrated, called the Theſis or Affirmation.

e. g. If a Right Line, cutting two Right Lines, makes equal Angles with them both, thoſe lines are parallel.

Here, the Hypotheſis is, if the Angles are equal; and, the Conſequence, that the Lines are parallel.

SUPPOSITION. In demonſtrating ſome Theorems, it is neceſſary to have recourſe, frequently, to ſuppoſe ſuch and ſuch things, which are not ſo in reality; by the abſurdity of the conſequences, ariſing from ſuch a ſuppoſition, a concluſion is drawn, and the Demonſtration is made evidently to appear.

Such kind of Demonſtration is called *reductio ad abſurdum*, i. e. proving it to be abſurd, or impoſſible to be on that ſuppoſition; which, not being direct and poſitive, is, to many, very unſatisfactory; yet, if rightly conſidered, is full, though not direct Demonſtration.

CONSTRUCTION, is the contriving or diſpoſing, geometrically, Lines and Figures, neceſſary for making the Demonſtration appear, clear and conſpicuous; and muſt always

always be made of ſuch Figures or Lines as are already well underſtood; the Properties, of which, being previouſly demonſtrated.

DEMONSTRATION. When any thing is propoſed, or affirmed in a Propoſition, the Caſe is firſt ſtated and prepared, by drawing ſuch lines, or forming ſuch a Conſtruction as is neceſſary; and it is afterwards demonſtrated; that is, the truth of the Aſſertion is made to appear, obvious, and without the leaſt doubt remaining; the performance or operation of which is called the Demonſtration.

The three laſt Terms being common words in the Engliſh Tongue, may, by ſome, be thought impertinent; but, notwithſtanding the common acceptation of them is almoſt univerſal, yet the application of them in Geometry requires to be explained.

DEFINITIONS.

Of the eſſential and operative TERMS.

The Terms of Art, to be defined in any Science, are Names, arbitrarily given, by the firſt Authors, or others, to certain Symbols, Figures, Marks or Characters, poſſeſſing certain properties or relations, in reſpect of figure, poſition, ſituation, &c.

The operative Terms are, generally, technical Words, peculiar to that Science, though perhaps applicable to others; which are not of common uſe in Language, or have a different ſignification.

The following Definitions are frequently referred to, hereafter, for illuſtratien or proof of what is advanced in the Propoſitions.

When any Figure, &c. which we are contemplating, is found to poſſeſs ſuch or ſuch properties, we affirm it to be ſuch a Figure, as anſwers to them; or, in contemplating any Figure, given in the Premiſes, we affirm that it has ſuch or ſuch properties, arbitrarily, by the Definition of it; and therefore, it requires no other Demonſtration.

DEF.

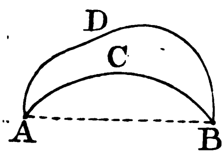
DEFINITIONS.

DEF. 1. A POINT is, rather to be conceived than underſtood to be, without dimenſions, therefore indiviſible.

In Plane Geometry, it is a Mark, ſuppoſed to be made with a ſharp-pointed Needle, or fine drawing Pen. As A.

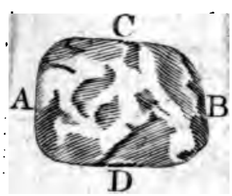
DEF. 2. A LINE has length only, without other dimenſions, of breadth or thickneſs.

DEF. 3. A RIGHT LINE is the ſhorteſt that can be drawn between two given Points, (A and B) uſually called a ſtreight Line.

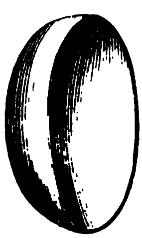


DEF. 4. A CURVE, or CURVED LINE, is any other than a Right Line; either regular, as ACB; or irregular, as ADB.

Curve Lines are of various kinds, as circular, elliptic, parabolic, &c. each of which has particular properties peculiar to them, and ſome in common.



DEF. 5. A SURFACE, or SUPERFICIES, is the outſide or external parts of Bodies, having no regard to thickneſs or ſubſtance; and therefore, has but two dimenſions, *viz.* length and breadth. As AB and CD.



Surfaces are various, as Convex, Concave, and Plane.

Irregular Surfaces are ſuch as are not, uniformly, any of the three, but may be compounded of them all.

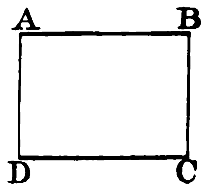
A convex Surface is one that is externally round or protuberant. Alſo, the outſide of the curve of a Circle, &c. is convex.

A concave Surface is one that is round internally, or hollow, ſuch as the inſide of a round Veſſel; the outſide is convex. The curve of a Circle towards the Center is concave.

DEF.

DEF. 6. A PLANE is a perfectly even and regular ſurface, which is neither convex nor concave, in any part; to which, if a Right Line or ſtreight Ruler be any how applied, it will touch in every Point.

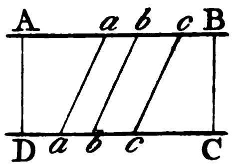
Or, if any two Points, in a Plane, be joined by a Right Line, the whole line is in that Plane.



A Plane may be conceived to be generated by the direct motion of a Right Line, laterally; or whirled around on any Point in it.

If the Right Line A B be moved, directly to CD, there will be generated the Plane A B C D.

N. B. A Plane may be of any Shape or Figure, in reſpect of its bounds or limits.

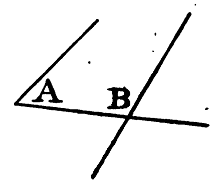


DEF. 7. PARALLEL. Right Lines are Parallel to each other, which, if produced infinitely, either way, and being in the ſame Plane, would never meet. As AB and CD.

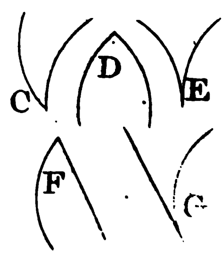
N. B. Parallel Lines are equidiſtant in every part; between which all Right Lines, as *a a*, *b b*, and *c c*, being alſo parallel, are equal.

But, the Diſtance between two parallel Lines is meaſured by a Perpendicular, A D or B C. See Perpendicular (Def. 10.)

The ſame holds true in parallel Planes.

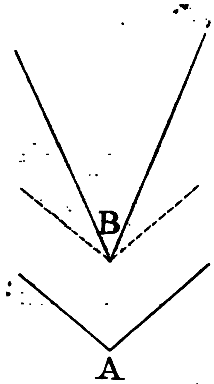


DEF. 8. A PLANE-ANGLE is formed by the meeting, touching, or cutting, of two Lines in the ſame Plane, ſo as not to fall into or conſtitute one Line.



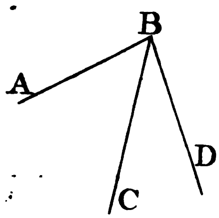
Angles are either right-lined, as A or B; curved, as C, D and E; or mixed, as F and G; which are compounded of a Right Line and a curved or crooked one.

N.B. Angles are neither increaſed nor diminiſhed by the length of the Lines which form them; for, the Angle A is greater than the Angle B, notwithſtanding the Lines which form the Angle B are longer than thoſe of the Angle A; but, the Lines forming the Angle B are more inclined to each other.



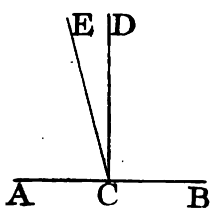
2. Imagine the dotted Lines to reprefent the Angle A laid upon, or applied to the Angle B; it is evident, that the Angle B is lefs than the Angle A, having a greater inclination of its Sides, or the aperture between them lefs; therefore, a lefs Angle may always be contained in a greater.

3. If the Sides of any Angle (as B) were lengthened infinitely, the Angle is not varied, if the inclination of the Lines remains the fame; but, if you fuppofe them moveable, on the angular point, like a folding Rule, and are parted farther afunder, approaching towards the dotted Lines, their inclination will then be lefs; and confequently, the aperture or opening between them greater; and therefore, the Angle is faid to be greater.



DEF. 9. VERTEX, is the angular Point in which two Lines, forming a Plane Angle, meet and touch each other.

N. B. A fingle Angle is ufually defcribed by one Letter only, or other Character, as A or B; but, if three or more Lines meet in a Point, then three Letters are ufed, to fpecify the Angle fpoken of; as ABC, CBD or ABD; of which the middle Letter always denotes the Vertex; and means, the Angle made by the lines AB and BC, or CB and BD, &c. wherefore, the middle Letter, denoting the Angle, is underftood to be twice named.



DEF. 10. PERPENDICULAR. A Right Line is perpendicular to another, when it does not incline to the other on either Side; but makes the Angle, on each Side, equal to the other.

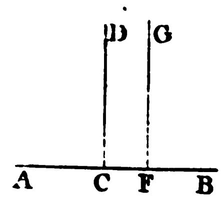
Thus; if the Angles ACD, DCB, are equal to one another, then CD is perpendicular to AB.

N. B. No more than one Perpendicular can be drawn from the fame Point in a Right Line, on the fame Side, and in the fame Plane.

For, if any other Line, as CE, be drawn from the fame Point, C, it cannot be a Perpendicular, but is faid to incline to AB; and, the inclination is on the fame Side of the Perpendicular CD; i. e. the Angle ACE is the Angle of Inclination of the two Lines AB and CE.

N. B. 2.

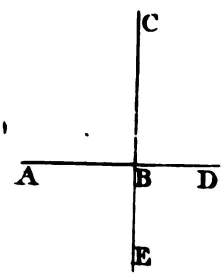
N.B. 2. If another Right Line, as FG, be drawn in the same Plane, from any Point F, perpendicular to AB, it will be parallel to CD.



DEF. 11. A RIGHT ANGLE is that which is formed by the meeting of two Right Lines which do not incline to each other, but either of them is perpendicular to the other.

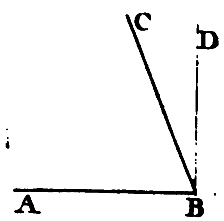
N. B. If either Side of a Right Angle be produced, or drawn out beyond the Vertex, there is necessarily generated another Right Angle. And, consequently, if both Sides are produced, there will be generated four Right Angles.

Thus, ABC is a Right Angle; if, when AB or CB is produced, towards D or E, there is made another, CBD or ABE; and if both are produced, EBD is a fourth Right Angle.



DEF. 12. An ACUTE ANGLE is less than a Right Angle.

If the Line CB, meeting AB in the Point B, falls on this Side of a Perpendicular, BD, at that Point; the Angle ABC, being less than the Right Angle ABD, is therefore called Acute.

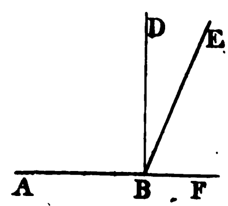


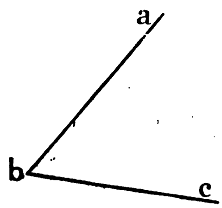
DEF. 13. An OBTUSE ANGLE is greater than a Right one.

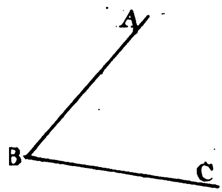
If the Line BE falls on the other Side of the Perpendicular, BD; the Angle ABE is Obtuse.

N. B. The difference, CBD, between an Acute Angle, ABC, and a Right one, ABD, is called the COMPLEMENT of the Angle ABC.

And, if either Side of an Obtuse Angle, as AB, be produced, the Angle EBF is the Complement of the Obtuse Angle; or its deficiency to two Right Angles, ABD, DBF.





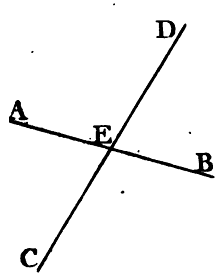


DEF. 14. EQUAL ANGLES. Angles are equal, when the Lines, which form them, have the ſame Inclination each to the other, reſpectively.

In the Angles a b c, ABC; if the Vertex, b, of the one, be applied to the Vertex, B, of the other, in ſuch wiſe, that the Side a b falling on A B, c b alſo falls on CB; then, there is the ſame inclination of c b to a b as of CB to AB, and the Angles a b c, ABC are equal.

N. B. The length of the Lines, or Sides, is not conſidered or regarded in the equality of the Angles, but only their inclination to each other.

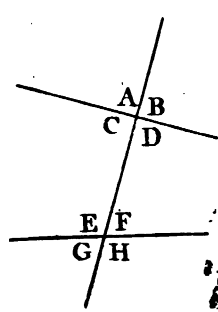
Angles have other Denominations, which are derived to them only from their Situation, in reſpect of each other; yet ſtill retain the general appellation of Right, Acute, or Obtuſe Angles. Such are the following.



DEF. 15. VERTICAL and CONTIGUOUS ANGLES.

If two Lines, AB and CD, cut and croſs each other, there are made four Angles, at the Point, E, of their mutual Interſection; either two of which, AED, CEB, or AEC and DEB, touching at their Vertices, only, are called Vertical Angles.

Any other two, as AEC, AED, or AEC and CEB, &c. having one Side, C E or A E, common to both Angles, are called Contiguous or Adjoining Angles.



DEF. 16. ALTERNATE ANGLES, and others.

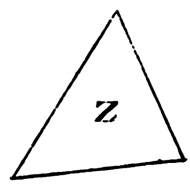
If a Line croſſes or interſects two Lines, there are made eight Angles, A, B, C, D, &c.; of which C and F, alſo E and D, between the two Lines, one on each Side of the cutting Line, are called Alternate Angles.

C and E, alſo D and F, are called INTERNAL ANGLES on the ſame Side.

E and A, F and B, C and G, or D and H are called INTERNAL and OPPOSITE ANGLES, on the ſame Side.

DEF.

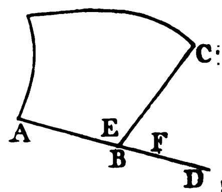
DEF. 17. A PLANE FIGURE is a Space bounded on all ſides by one or more Lines in the ſame Plane.



2. Any Line, between two adjacent Angles, forming or bounding a Figure, is called a SIDE of that Figure.

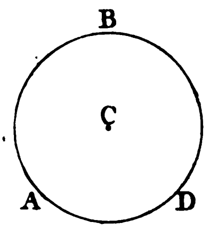
N. B. If a Plane Figure be bounded by Right Lines, only, it is called a Right-lined Figure; as Z. And if it be formed of Right Lines and curved, it is called a mixed Figure. As A C.

DEF. 18. INTERNAL and EXTERNAL ANGLES.



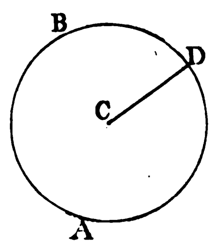
If any Side of a Plane Figure be drawn out beyond the Figure, as A B to D, the Angle E, or ABC, within the Figure, is Internal; and the Angle F or CBD, without the Figure, is called an External Angle.

DEF. 19. A CIRCLE is the ſimpleſt and moſt perfect of all Plane Figures, therefore the firſt; it is bounded by one regular and uniform curved Line, falling again into itſelf; which is called the CIRCUMFERENCE of the Circle.



The curved Line ABD is the Circumference; the Space, included within it, is the Circle.

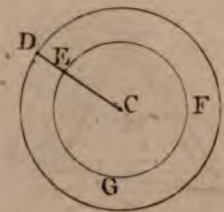
DEF. 20. CENTER of a Circle, or CENTRE, is a Point in the middle of a Circle, or the middle Point of a Circle; which is equally diſtant, every way, from the Circumference. As C.



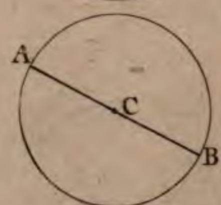
N. B. The geneſis of a Circle is thus defined.

If a Line, CD, be conceived to be revolved quite around, on one extreme, C, fixed to a Pin or Point; the other extreme, D, will, in its revolution, deſcribe the Circumference of a Circle, ABD; and the Line CD, having gone over the whole ſpace, has generated the Circle, bounded by that Circumference.

Hence, it is evident, that all Right Lines, drawn from the Center of a Circle to the Circumference, are equal.



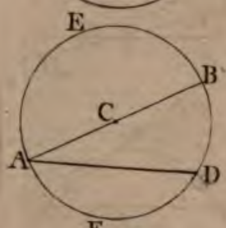
N. B. 2. Equal Right Lines deſcribe equal Circles; but, it another Line, CE, or, if any Point, E, be aſſumed in the Line CD, another Circle will be deſcribed, on the ſame Center, and in the ſame ſpace of time; whoſe Circumference, EFG, is to that of the other Circle, deſcribed by the Point D, in proportion to the Lines CE and CD, by which they were generated.



DEF. 21. DIAMETER, of a Circle, is a Right Line drawn through the Center, and terminated on both ſides by the Circumference. As A B.

2. Half the Diameter of a Circle is called the Radius. As AC or CB.

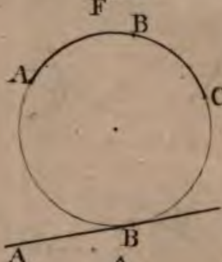
N. B. Every Diameter divides the Circle, and alſo the Circumference, in two equal Parts.



DEF. 22. A SEGMENT of a Circle, is any portion cut off by a Right Line; which is called a Chord or Subtense.

As AD, making two Segments, AED and AFD.

N. B. A Diameter is alſo a Chord Line.



DEF. 23. A SEMICIRCLE is a Segment, made by a Diameter, AB. As AEB or AFB.

Therefore, the Segment AED, which is greater than a Semicircle, is called a greater Segment; and AFD a leſſer Segment.

DEF. 24. An ARK, or ARCH, is any portion of the Circumference of a Circle. As AB, BC, or ABC.

DEF. 25. A TANGENT is a Right Line drawn without a Circle, and touching it in a Point only; which is called the Point of Contact.

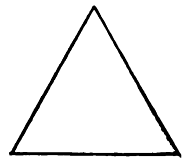
As AB, touching the Circle in B.

DEF. 26. A TRIANGLE is a Plane Figure bounded by three Right Lines, and contains as many Angles.

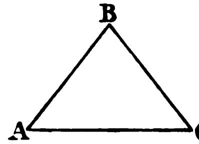
N. B. Not

N. B. Not leſs than three Right Lines can include a Space and form a Figure; wherefore, a Triangle is the firſt of all Right-lined Figures.

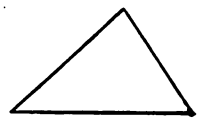
Triangles are of various kinds. As follows.



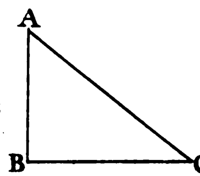
DEF. 27. 1. An EQUILATERAL TRIANGLE has all its three Sides equal, to one another.



DEF. 28. 2. An ISOSCELES TRIANGLE has only two equal Sides. AB and BC.

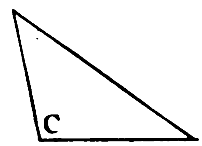


DEF. 29. 3. A SCALENE TRIANGLE has all its Sides unequal.



DEF. 30. 4. A RIGHT-ANGLED TRIANGLE is one that has a Right Angle. B.

2. The Side A C, oppoſite the Right Angle, is called the HYPOTHENUSE.



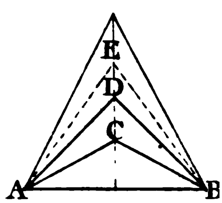
DEF. 31. 5. An OBTUSE ANGLED TRIANGLE is one which has an Obtuſe Angle. C.

N. B. The two laſt are not diſtinct ſpecies of Triangles, but only a particular kind; which ſtill come under the general Denomination of Iſoſceles or Scalene.

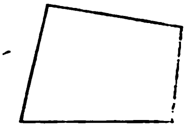
An Iſoſceles or Scalene Triangle may be either right or obtuſe angled, or have all its Angles acute.

The Triangle ADB is right-angled; ACB is obtuſe-angled; and AEB has all its Angles acute; yet, they are all Iſoſceles.

So likewiſe, the Figures of the two laſt Definitions are Scalene Triangles.

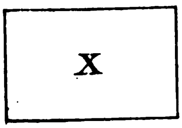


DEF. 32. A QUADRILATERAL or QUADRANGLE is a Plane Figure which has four Sides, and four Angles.



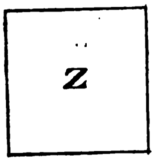
Theſe are ſynonimous Terms; the firſt expreſſing it by the number of its Sides, the other by its Angles.

DEF. 33.



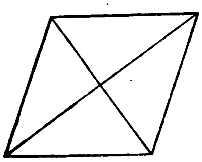
DEF. 33. A PARALLELOGRAM is a Quadrilateral, whofe oppofite Sides are parallel.

DEF. 34. A RECTANGLE is a Parallelogram, whofe Angles are all Right ones. As X.



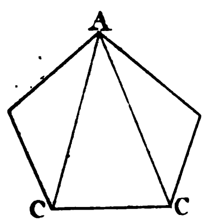
DEF. 35. A SQUARE is a Rectangle, whofe Sides are all equal, to one another. Z.

N.B. All Rectangles and Squares are Parallelograms.

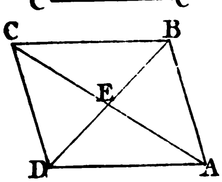


DEF. 36. A RHOMBUS is a Parallelogram, whofe Sides are all equal, and its Angles not Right ones.

2. If the Sides of a Parallelogram are not all equal, and the Angles not Right ones, it is called a RHOMBOIDES.

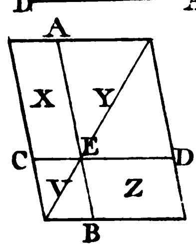


DEF. 37. A DIAGONAL is a Right Line drawn between any two Angles, that are oppofite, in any right-lined or mixed Figure; i. e. from one Angle to the other. As AC.



N. B. In Parallelograms, the Diagonal is ufually called a Diameter; becaufe it paffes through the Center (the middle Point, E, where the two Diagonals, AC and BD, interfect) and, as in a Circle, it divides the Parallelogram into two equal Parts.

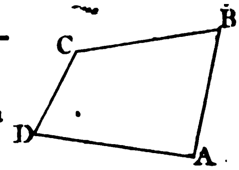
Any Right Line, cutting a Parallelogram through its Center, is a Diameter.



DEF. 38. COMPLEMENTS of a Parallelogram.

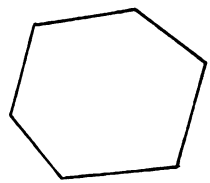
If any Point, as E, be taken in the Diagonal of a Parallelogram, and, through that Point, two Right Lines are drawn parallel to the Sides, both ways (AB and CD) it will be divided into four Parallelograms, V, X, Y, and Z; the two, X and Z, which touch the Diameter, in the Point E, only, are called the COMPLEMENTS; which, with either of the other, about the Diameter, taken together (as XYZ or XVZ) is called a GNOMON.

DEF. 39.

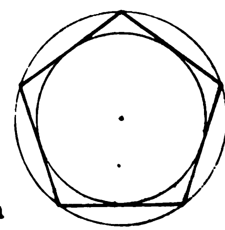


DEF. 39. A TRAPEZIUM is an irregular four-sided Figure.

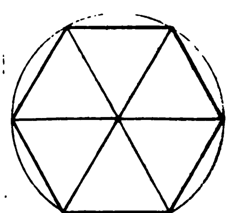
Wherefore, every Quadrilateral which is not a Parallelogram, is, consequently, a Trapezium. As ABCD.



DEF. 40. A POLIGON. All right-lined or other Plane Figures, having more than four Sides, have the general appellation of Poligons, signifying many Sides.

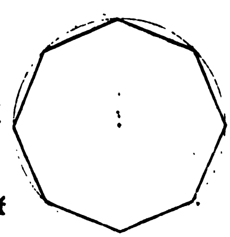


N. B. Ordinate or regular Poligons' are such as have all their Sides and Angles equal; about which a Circle may be circumscribed, whose Circumference shall pass through every Angle of the Poligon; and, a Circle may also be inscribed, which shall touch every Side.



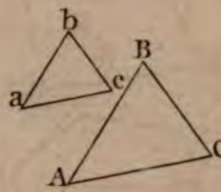
Poligons have various Names, derived to them from the Number of their Sides; as follows.

1. A PENTAGON is one that has five Sides.
2. A HEXAGON has six Sides.
3. A HEPTAGON has seven Sides.
4. An OCTAGON has eight Sides.
5. A NONAGON has nine Sides.
6. A DECAGON has ten Sides.
7. A DUODECAGON has twelve Sides.
8. A QUINDECAGON has fifteen Sides.



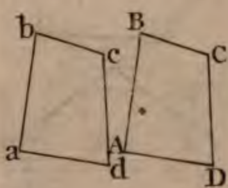
These eight are the most essential, in Geometry, and the most useful amongst Mechanics. To specify every kind of Poligon would be infinite.

DEF. 41. PERIMETER is the sum or measure of all the Sides of a Poligon, or other right-lined or mixed Figure, in one Sum; which is sometimes called its Circumference, or PERIPHERY.



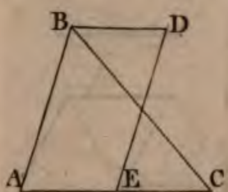
DEF. 42. EQUIANGULAR FIGURES are ſuch as have an equal number of Angles; and the Angles of one alſo equal, reſpectively, to the Angles of the other, each to its correſponding Angle.

As, a equal A, b equal B, and c equal C.



DEF. 43. CONGRUOUS FIGURES are ſuch as have all their Angles equal, reſpectively; alſo, the Sides, which contain equal Angles, or which are between equal Angles, are equal.

If the Angle a be equal A, b equal B, c equal C, and d equal D; and, if the Side ab is equal AB, bc equal BC, cd equal CD, and ad to AD; then, the Figures abcd, ABCD, are congruous.



DEF. 44. EQUAL FIGURES are ſuch as have an equal Area. See Def. 47.

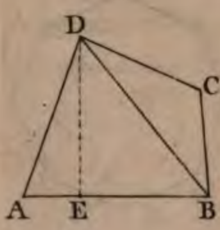
N. B. If two Figures of different Denominations have an equal Area, they are called equal Figures.

So, the Parallelogram ABDE is equal to the Triangle ABC.

Congruous Figures are equal and ſimilar.

DEF. 45. BASE, of a Plane Figure, is the Side on which it is ſuppoſed to ſtand erect. As AB.

N. B. It may be any Side, at diſcretion; but is generally applied to the lower Side, or that which is next towards us.



DEF. 46. ALTITUDE, of a Figure, is its perpendicular height from the Baſe.

As ED, perpendicular to the Baſe AB, is the Altitude of the Trapezium ADCB; or of the Triangle ADB.

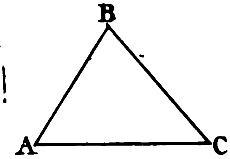
DEF. 47. AREA of a Plane Figure, or other Surface, is its ſuperficial Contents; i. e. the meaſure, or quantity of Space contained within the bounds of the Figure, expreſſed in ſquare Feet, Yards, or any other known meaſure, of length.

DEF. 48.

DEF. 48. SEGMENT of a Line, is any portion of a Line.

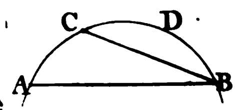
As AC, or CB, of the Line AB.

DEF. 49. SUBTEND. The Side of a Triangle which is oppofite to any Angle is faid to fubtend that Angle.

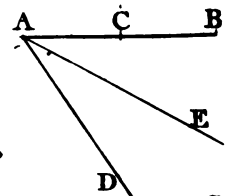


Thus, AB fubtends the Angle C, AC fubtends the Angle B, and BC fubtends the Angle A.

So likewife, the Chord or Subtenfe AB, fubtends the Ark ACDB; and CB fubtends the Ark CDB.



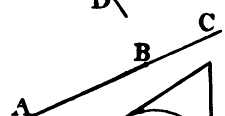
DEF. 50. To BISECT, is to cut or divide a Line or Angle, &c. into two equal Parts.



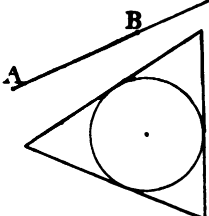
Thus AB is bifected in the Point C; and the Angle BAD, is bifected by the Line AE.

2. To TRISECT, is to cut, equally, into three Parts.

DEF. 51. To PRODUCE, is to draw out a Line or Plane, or to lengthen it at pleafure.

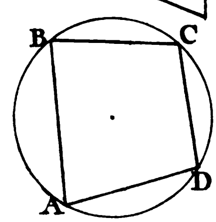


Thus, the Line AB is produced to C.



DEF. 52. To DESCRIBE, is to draw a Line, Circle, or other Figure.

DEF. 53. To INSCRIBE, is to draw a Figure touching every Side of another Figure, internally; or, whofe Angles fhall all touch the Circumference of a Circle.



DEF. 54. To CIRCUMSCRIBE, is to defcribe a Circle or other Figure paffing through all the Angles of another Figure. As ABCD.

DEF. 55. EXTREMES or BOUNDS, are the utmoſt Limits of any thing.

So, the extremes of a Line are Points; the bounds or extremes of a Surface are Lines, (except the ſurface of a Sphere, which has no Bounds) and, the extremes of a Solid are Surfaces.

A Sphere has but one Surface; alſo, a Circle is bounded but by one Line, without beginning or end; conſequently it has no Extremes.

Some Readers will, perhaps, think I have already deviated from the Plan propoſed in the Preface, which was, to abridge the Elements of Euclid; but I hope they will find, that the time ſpent on the Definitions (by way of Introduction) is not thrown away; having made ſeveral of them uſeful Leſſons, as well as defined the Terms; and I am perſuaded, that the more perfectly they are underſtood, the progreſs, in the Subject they relate to, will be greatly facilitated. But, I muſt inform the Reader, that it is not the length of the Definition itſelf, but the number of Terms I have defined, and the Notes to ſeveral, which have ſwelled the bulk of them; having near twice the Number which ſome Authors have, and, in my opinion, not one ſuperfluous.

However, not to diſcourage the young Student, I adviſe him not to burthen his memory too much at once; for there is not the leaſt neceſſity that he ſhould retain them all, before he proceeds further; 'tis enough, at firſt, to read them over with attention, and underſtand them clearly as he reads; they will ſoon become familiar to him, by frequent uſe, in the courſe of his ſtudy of the Science.

I alſo particularly adviſe him, when he meets with any Term hereafter, of which he has not a clear Idea, to turn immediately back to the Definition of it; he may depend on it, that the time will not be entirely loſt.

Several of the Terms which I have here defined, being frequently uſed in the Subject of Geometry, are abſolutely neceſſary to be defined, but are not, properly ſpeaking, elementary. Such are the 4th, the 9th, the 14th, 15th, 16th, and 18th; the 36th, the 41ſt, 43d, 44th, 47th, and 48th, and the laſt ſeven; which are chiefly operative or practical Terms. Though I muſt freely own, that I ſee no reaſon why the 49th, 50th, 51ſt, and 52d, are not as neceſſity to be defined as the 53d and 54th: the 55th is made three ſeparate Definitions, in the 3d, 6th, and 13th of Euclid.

It is, or ought to be, the deſign of every Author on any Science, to make his Book a perfect Tutor; conſequently, no particular Term, made uſe of in that Science, and which is peculiar

to

to it, should be left undefined: for, we should suppose the Student to be entirely ignorant of all that relates to it. Can any thing be more absurd, than to propose bisecting a Line or Angle, or producing a Line, &c. to talk of Alternate Angles, &c. of Diagonals, Complements, Subtenses, &c. without having first defined what is meant by them? I have often been surprized at the omission of the Definition of a Poligon, and the various kinds of Poligons; the forming, inscribing, and circumscribing of which, is the principal subject of the 4th Book. They are frequently called by the Terms Pentagon, Hexagon, &c. which have never been properly, if at all, defined. In the 23d Definition they are, in general, called many-sided Figures; 'tis a strange ungeometrical Term, and never once made use of after, but other Terms are assumed.

Euclid, himself, has not defined a Parallelogram, that most useful and necessary Figure. I admire Mr. Stone's Apology for that omission, as being unwilling to increase the Number, unnecessarily; and says, that Euclid has sufficiently defined it in the 34th Proposition. Mr. Stone indeed has, but not Euclid; for why else has Keill defined it before the Proposition? But, if it be thought most eligible to define that Term in the Proposition, where it is first made use of, why are not others so defined? why are not all, as well as this?

At the same time, Euclid has, in Mr. Stone's Opinion, given several unnecessary Definitions; viz. the 3d, 6th, and 13th of his Euclid, which I always thought superfluous; the 9th I think so too, also the 20th. He, likewise, thinks the 18th, 26th, the 32d, and 33d, useless, In respect of the 26th (the 29th of this) I cannot say it is essentially necessary; yet, as Scalene Triangle is a general Term, including all that are neither Equilateral nor Isosceles, I cannot think it redundant.

The 29th, of which he says nothing, is really superfluous, viz. "when all the Angles are acute, it is called an acute-angled Triangle." I know of no properties peculiar to such an acute-angled Triangle, that is not common to every Triangle; for every Triangle has, necessarily, two acute Angles. An equilateral Triangle is included in that Definition, and so are the Isosceles and Scalene, frequently.

Why is not a three-sided Figure (Def. 21.) called a Triangle, as it is always called afterwards? we might as well go on, from four to five or fifteen sided Figures. An Oblong (Def. 31.) for a Rectangle, is quite ungeometrical, and is never called so after. The Rhombus and Rhomboides, 32. and 33. (Def. 36. of this) are, in a great measure, useless, being fully signified in a Parallelogram, which also includes Squares and other Rectangles; all which, are only particular species of Parallelograms; the Rhomboides including all unequal sided and acute angled.

I can by no means agree with Mr. Stone, in thinking the 18th Definition, of a Semicircle, needlefs; I alfo think, the Radius as neceffary to be defined as the Diameter. In fpeaking of Angles in a Segment of a Circle, Prop. 31. of the 3d Book, the 12. of this, viz. "the Angle in a Semicircle is a Right one," &c.; fince there is a neceffity for calling it fomething, I cannot but think a Semicircle more elegant and expreffive than half a Circle, which does not confine it to any particular Shape or Figure; provided it has half the Area, it is half a Circle; whereas, the Semicircle (which undoubtedly means half a Circle) is always underftood to be contained under a Diameter, and half the Circumference.

Mr. Stone is fomewhat too dogmatical in his remarks. I cannot be of Opinion, that the manner in which Euclid has compiled his Elements, is the beft that can poffibly be, becaufe Euclid lived two thoufand Years ago. 'Tis not to be imagined that the mathematical Sciences had arrived at their *ne plus ultra* in his Days, for it is notorious they were not; why then, fhould we fuppofe, the Elements of Geometry to be in their greateft perfection? I am not fo much wedded to antiquity, as to think them infallible; neither do I think that all Euclid's Definitions are neceffary, or that he has omitted none that are fo: Such, therefore, as are ufelefs I have rejected, and have fupplied their places with others, which I think more effential, and abfolutely neceffary to be defined.

POSTU-

POSTULATES.

Poſtulates are fundamental Principles in any Science; which, being plain and ſimple in themſelves, may readily be granted; although, it is not poſſible to give perfect and indiſputable Demonſtration.

The following are, therefore, requeſted to be granted.

1. That there may be a perfect Plane; and that it may be extended at pleaſure.
2. That a Right Line may be drawn between any two given Points; i. e. from one Point to the other.
3. That a Right Line may be produced at pleaſure; i. e. that a finite Line may be continued or lengthened.
4. That a Circle may be deſcribed on any Center, and with any given Radius.
5. That one Figure may be applied to, or laid upon another Figure.

Theſe Poſtulates muſt be granted, at leaſt in Idea, or all Geometry falls at once to the Ground; for, if there cannot be a Plane, a Right Line, or a Circle, the whole Elements of Geometry are to no purpoſe; as it will be impoſſible to form a Conſtruction, whereby we may demonſtrate the moſt eſſential Properties of Figures in general, whether Plane Figures or Solids; and conſequently, if no Demonſtration can be given, there is an end of the Science, having no Data to build on.

In every mathematical or phyſical Science, there is a neceſſity for ſome Data or firſt Principles to be given, whereon to frame Hypotheſes, in order to demonſtrate the Theorems which follow; and the more ſimple thoſe Principles are, the better; becauſe there will be leſs room to diſpute them. But, at the ſame time, if they are diſputed, they are the moſt difficult to demonſtrate, for being the moſt ſimple; becauſe, there is no reverting back to any thing more ſo; and conſequently, there can be no Demonſtration given. That the thing is ſo, of itſelf, is ſomewhat arbitrary, notwithſtanding there is no poſſibility of denying it; therefore, the more ſimple the firſt Principles are, the readier the aſſent will be given; and the Demonſtrations, of the moſt complex Propoſitions, which follow after, will be eaſier obtained and more firmly ſupported; and conſequently, the whole Science, which is built on thoſe Principles, is more ſolid and permanent, and more ſecurely eſtabliſhed.

THE

THE

THEORY

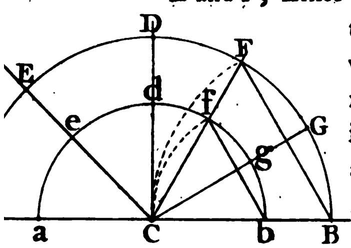
OF

PLANE ANGLES.

BEfore I proceed to practical Geometry, I think it proper, first to explain the Theory of Plane Angles; which I look on as a very necessary Introduction.

In order to have a perfect and clear Idea of Plane Angles, and to determine their Quantity by a certain standard or measure, the Circumference of a Circle is supposed to be divided into 360 equal parts, called Degrees (answering to the number of the Degrees on the Equator) it is evident that those divisions will be less in a small Circle than in a larger.

1. If from two Points in the Circumference of a Circle, as E and F, Lines are drawn to the Center, as EC and FC, there is made an Angle at the Center, C, which is greater or less, according to the number of Degrees on the Ark EDF, but it will be the same in a small Circle as in a large one, i. e. the lines will have the same inclination to each other. e. g.



2. A D B is a Semicircle, whose Center is C; the Arch AEDFB contains 180 deg. half 360, the whole Circumference.

From the middle point, D, of the Arch ADB, which is 90 deg. each way, from A and B, if CD be drawn, it will be perpendicular to AB; for ACD and DCB, are Right

Right Angles, at the Center (Def. 11.) the Arks AD, DB, being each a fourth part of the whole Circumference, or half the Semi-circumference: hence, a Right Angle is ſaid to be of 90 Degrees.

3. If the Ark AD be biſected in E, and EC be drawn, the Angles ACE, ECD, will be each of 45 deg. half ACD a Right Angle, or 90 Degrees.

And, if the Ark DB be triſected, at F & G, (i. e. divided into three equal parts), and FC be drawn, the Ark DF containing 30, and FGB 60 deg. the Angle DCF is ſaid to be an Angle of 30, and FCB an Angle of 60 deg.

By which means, an Angle of any quantity may be obtained, or meaſured.

4. On the ſame Center, C, with any other Radius, as C a, let the Arch a e d f b be drawn, which is alſo a Semicircle.

It is very obvious, that it is alſo divided into the ſame number of parts, and in the ſame proportion, as the Arch AEDFB; for it is biſected in d, and a d is again biſected in e, and d b is alſo triſected at f and g; wherefore, AD, a d are each a fourth; ED, e d an eighth; BF, bf, a ſixth; and FD, f d, a twelfth part of their reſpective Circles; and the Angles ACD, aCd; ECD, eCd, &c. are the ſame in both.

From all which, it is clear, that, Angles may be formed or meaſured by an Ark or Circle of any Radius. And alſo, that equal Arks of the ſame, or of equal Circles, or an equal number of degrees in a Circle of any Radius, will form equal Angles at the Center.

5. If you would have an Angle of 60 degrees at the point C, of the line BC.

With

With any Radius, at pleaſure, deſcribe the Ark BD, cutting the Line BC in B; with the ſame Radius, on the Center B, cut the Ark BD at F, and draw CF.

It is very clear, that the Angle BCF would be the ſame, if a leſs Radius had been taken, as C b.

For, draw the Chord Lines F B and f b, each will be equal to the Radius of its reſpective Circle; and, the Triangles CFB, C f b are equilateral; whoſe Angles are of 60 degrees each (Cor. 1. 9. 1.); for the Arks BF and b f are, each a ſixth part of the whole Circumference of their reſpective Circles, of which C is the common Center, (ſee Prop. 11. 4.); and conſequently, each contains 60 degrees on the circumference of that Circle, of which it is a Part; whoſe Radius is CB or C b.

A deſcription of the Inſtrument called a PROTRACTOR, with the application of it, in meaſuring and making Angles, of any known quantity or meaſure, may not be improper in this place. It is of ſpecial uſe in Surveying, in drawing Plans of any piece of Ground for building on, &c. or of Buildings, already erected, being readier and more exact than a Line of Chords.

The Protractor is a Semicircle, divided on its Limb or Semi-circumference, AEDFB, into 180 equal parts; having a ſmall Notch at C, the Center. See the laſt Figure.

Some Protractors have a Scale, added to the Semicircle, which are the beſt and readieſt in uſe.

In meaſuring an Angle, apply the Diameter, i. e. the Edge or Right Line AB, to either Side of the Angle ACE, with the Vertex, C, of the Angle, at the center of the Protractor; and, where-ever the Side CE, cuts the Limb or circular edge of the Inſtrument, obſerve how many Degrees there are from A to E, the Ark intercepted between the Sides AC and CE, of the Angle ACE.

If

If it contains 45 or 50, or whatever number of Degrees it happens to be, (as 45 by the Figure) the Angle ACE is of ſo many Degrees. If it had cut the Arch at D, as ACD, it is a Right Angle; and if beyond D, as ACF, it is obtuſe; the Complement FCB, i. e. the Ark FB, being ſubtracted from the Arch of the Semicircle, ADB, or 180 degrees, gives the quantity of the Angle ACF.

2. If it is required to lie down or make an Angle of ſome known Quantity (as 45 deg.) at the Point C, of the Line AB.

Apply the edge of the Protractor, as above, with the Center, C, at the Point given; make a Mark or Point at E; take away the Inſtrument, and draw E C.

Thus, may any Angle whatever be laid down on Paper.

A Scale of equal Parts is nothing more than a Right Line divided into any number of equal Parts, at pleaſure.

Each Part may repreſent any meaſure you pleaſe, as an Inch, a Foot, a Yard, &c.; for, being equal, whatever meaſure any Object or Figure contains, in length and breadth, a ſimilar Figure may be conſtructed, on a Plane, having or containing the ſame number of Diviſions, each way, on the Scale, as the real Object contains of Feet, Yards, &c. One of thoſe Parts is generally ſubdivided into parts of the next inferior denomination, or into tenths and hundredths, denoting the Decimal Parts.

Obſerve, that the Diviſions on the Scale, (whatever meaſure is repreſented by them,) muſt always be adapted to the Proportion you would delineate any Object, or form a Deſign.

See the Appendix (Page 15 and 16) for the conſtruction of Scales.

N. B. A pair of Compaſſes or Dividers, a Drawing Pen, and a ſtreight Ruler, are all the Utenſils that are requiſite, in Plane Geometry.

The Board or Paper, on which we draw any geometrical Figure, is ſuppoſed to be a Plane.

ABBREVIATIONS &c. Explained.

&c. *Et cætera.* And all the reſt. When, what is ſuppoſed to follow may be readily underſtood.

i. e. *Id eſt.* That is. When, what has been ſaid requires to be further explained.

viz. *Videlicet.* To wit; or, that is to ſay. When any thing advanced is given in Groſs, which is more particularly ſpecified, as follows after.

e. g. *Exempli gratia.* For inſtance; or, for the ſake of example. When an Example is to be given of what is advanced.

N. B. *Nota Bene.* Mark well. That is, take particular notice of that Paragraph.

Q. E. F. *Quod erat faciendum.* Which was required to be done.

Q. E. D. *Quod erat demonſtrandum.* Which was to be demonſtrated.

DEM.	Demonſtration.	Hyp. AB.	Hypothenuſe AB.
COR.	Corollary.	Perp.	Perpendicular.
SCHOL.	Scholium, or remark.	Dia.	Diameter.
APPL.	Application, or uſe.	Diag.	Diagonal.
Par. AB. or P. AB.	Parallelogram AB.	Trap.	Trapezium.
		Pent.	Pentagon, &c.

Rect. AB. Rectangle AB, or ABCD, &c.

R. Ang. ABC. Right Angle ABC, &c.

Line AB. Denotes Right Line AB, &c.

Note. When it is required to join any two Points, it is meant, that a Right Line be drawn between them, i. e. from one Point to the other.

AB-

ABBREVIATIONS, by way of Reference.

Poſt. 1. or 2.	Refers to the firſt or ſecond Poſtulate; where it is requeſted that ſuch or ſuch things may be granted.
Def. 6. or 7. Or, Def. 6. 3. or 6. 5. &c.	Refers to the ſixth or ſeventh Definition in the general Introduction. But, when there are two numbers, it refers to the 6th Definition of the 3d or 5th Book.
Ax. 3.	Refers to the third Axiom for Demonſtration, ariſing from ſelf-evident properties of things.
2. Th. P. A.	Refers to the ſecond Article in the Theory of Plane Angles, for illuſtration or Proof.
Pr. 1. or 2. &c.	Refers to the firſt or ſecond Problem, for the conſtructing of ſome Figure, &c.
P. 1. or 2. &c.	Refers to the firſt or ſecond Propoſition of that Book, for proof of the Aſſertion.
P. 2. 3. &c.	To the ſecond Propoſition of the third Book.
C. 2. 4. 1.	To the 2d Corollary, of the 4th Propoſition of the firſt Book.
Hyp.	That the thing is ſo by the Hypotheſis, or is given in the Premiſes.
Sup.	That it is ſo by Suppoſition, only.
Con.	That it is ſo by Conſtruction; i. e. the thing was formed or made ſo.

Conſ. conſequently. Th. therefore. Wh. wherefore, it is ſo.

Note. When there is but one Number within Parentheſis or otherwiſe, as (15.) &c. it refers to the fifteenth Problem or Theorem of that ſame Book. But if there be two Numbers, as (10. 1.) or (12. 3.) &c. it refers to the 10th Prop. of the firſt Book, or to the 12th of the third, &c.

INSTRUC-

INSTRUCTIONS

FOR

YOUNG STUDENTS.

I would advise the young Practitioner to draw every Figure as he proceeds; carefully remarking what things, as Points, Lines, Angles, &c. are given; which are, in general, stronger marked than the operative Lines; they being either dotted or finer drawn; the given Lines, &c. are, by that means, obvious and distinguishable from the other.

In Geometry, as in Arithmetic, there is always some Data or things given; from which, in Theory, other Properties are deduced, as a necessary consequence; and, in Practice, somewhat is required to be done, or performed, from what is given.

Let the Practitioner, therefore, select the given things, and mark them down, first, in the position given in the Premises; but, with as much variation as it will admit of; i, e. he need not put them exactly as in the figure, only observe that they are as required.

e. g. In Prob. 4. an Angle is required to be made at the extremity of a given Line; but the position of that given Line is not absolutely determined; also, the Angle may be made at either extreme, and either above or below the given Line.

Likewise, in the 6th and 7th Problems, the Perpendicular may be drawn, and the Point, in the 7th, given on either Side of the Line; for, let it be observed, and carefully remembered, that, by the Term Perpendicular, nothing more is meant than the Position one Line has to another; which Position, is when they make a Right Angle or Right Angles with each other; no regard being had to the position or situation of either, separately.

These things being premised, and the given Lines, &c. described on Paper; carefully observe the directions given in the operation, and proceed accordingly, step by step, drawing every Line, Angle, Ark, &c. as the Problem directs.

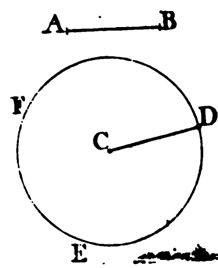
PRAC-

PRACTICAL GEOMETRY.

PROBLEM I.

To describe a Circle of any given Radius, and on a given Center.

AB is the Radius given, and C the given Center.



FIX one Point of a pair of Compasses in either extreme of the given Line, AB, and extend the other Point to the other extreme, i. e. open the Compasses equal to the given line.

Then, fix one Point of the Compasses in C, the Center given, and revolve the other Point around; which, by its revolution, will describe the Circumference DEF. - - - - - Post. 4.

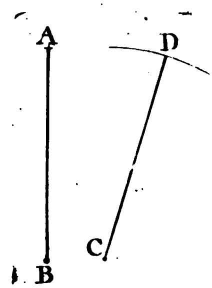
DEF contains the Circle required (Def. 19.)

For, if a Right Line be drawn from the Center to the Circumference, as C D, it is equal to the given Line A B; by Construction.

PRO-

PROBLEM II. 2nd. I. Euclid.

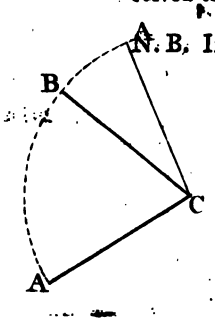
To draw Right Line from a Point given, C, which shall be equal to a given Line; AB.



Extend or open the Compasses equal to the Line given; fix one of their Points in C, the given Point, and, with the other, describe a Circle, or portion of a Circle only, at D. (Pr. 1.)

Apply a streight Ruler close to the Point C, and crossing the Ark at D, (in whatever position you require the Line to be) and, with the point of a Pencil or a Drawing Pen (applied first to the Point C, and drawn, along the edge of the Ruler, to the Ark at D) describe the Right Line CD. (Post. 2.) Which is equal to the given Line AB; by Construction; and by N. B. Def. 20.

SCHOL. Thus, the genesis of a Right Line (Def. 3.) is conceived to be by the direct motion of a Point.



N. B. In the practice of Geometry, it is often required to draw or to make a Line equal to another Line given; which is done by drawing an Ark of a Circle, as AB, from the given Line, AC, till it cuts the other; if the two Lines, AC and CB, touch at the Point given, C, which is made the Center.

But, if they do not touch, the Line given is taken for Radius (as in the Problem) and an Ark drawn where it is required; for equal Circles have equal Radii, as well as all Radii of the same Circle are equal; which needs no other Demonstration than the genesis of a Circle, in N. B. Def. 20.

So that, hereafter, when two Lines are observed to be Radii of the same Circle, it is sufficient Demonstration that the Lines are equal; and also, when they are made Radii of equal Circles.

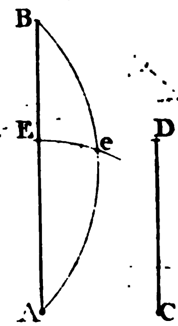
APPL. The Application of this Problem, in designing, is to delineate or draw, on Paper, &c. a Right Line equal to some known measure, as AB, by a Scale of equal Parts.

PRO-

PROBLEM III. 3. I. Euclid.

A Right Line being given, to cut off a portion or ſegment equal to another given Line, or known meaſure.

AB is the firſt given Line, and CD the meaſure of the Segment required to be cut off.



Extend the Compaſſes from C to D, i. e. with the Radius CD, ſetting one Point of the Compaſſes in either extreme of AB, (from which the given Segment is required to be cut off), as A, with the other Point, draw a ſmall Ark cutting the Line AB, at E. Q. E. F.

2. After the ſame manner, any portion of an Ark of a Circle may be cut off. As AeB.

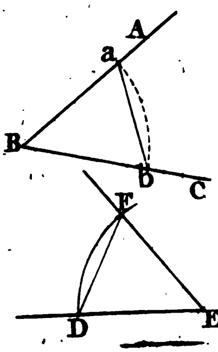
APPL. In delineating, the given Line AB may be ſuppoſed to be an indefinite Line already drawn; or, it may repreſent a certain meaſure, by ſome Scale, ſuppoſe 5 Feet, being made equal to 5 Diviſions on the Scale; and it is required to cut off 3 Feet from the extreme Point A, of the Line AB, or two Feet from the other extreme, B.

By which means, a Right Line may be divided in any Proportion required.

PROBLEM IV. 23. I. Euclid.

To make an Angle, equal to a right-lined Plane Angle given.

ABC is the given Angle, and DE a Line given; it is required to make, at the point E, an Angle, with the Line DE, equal to the given Angle.



With any Radius, at discretion, on the Vertex of the given Angle, B, describe an Ark, a b, cutting the two Sides AB and BC in the Points a and b. - - - - - Post. 4.

With the same Radius, on the Point E, draw the Ark DF; take the measure of the Ark ab in your Compasses; make DF equal ab. - - 2. Pr. 3. Draw EF through the Point F; and it is done; Q. E. F.

The Angle DEF is equal to the given Angle ABC; i. e. FE inclines the same to ED, as AB to BC.

This is evident, from the Theory of Plane Angles, Art. 4. ab and DF being equal portions of equal Circles.

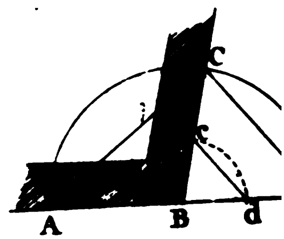
Or, by drawing the Chord Lines ab and DF, the Triangles Bab, EDF, are Congruous. - - - - Con. And the Angle DEF is equal ABC. - - P. 7. 1.

APPL. From this Problem we learn to delineate, i. e. to lay down or draw, on Paper, any right-lined Angle, of a piece of Ground or Building, &c. which we have measured, to form a Plan of it.

The

The Angle may be taken in the manner following.

If it be an internal Angle, as CBD; set off equal measure, from B, to C and D, and draw, or measure only, the Diagonal, or Chord CD.



Suppose you have made BC and BD each equal 5 feet, and the Chord CD measures 6 feet 6 inches, or as it happens; then, by a Scale of equal Parts, having drawn BD, at discretion, take 5 divisions off the Scale; which may be either inches, half inches, or any other measure, provided they are equal.

With that Radius, on B, draw the Ark CD, cuting BD in D.

Take $6\frac{1}{2}$ of the same divisions in your Compasses; and, seting one Point in D, make a small Ark, at C, with the other, and draw BC.

So shall CBD be an Angle laid down, equal to the Angle which was measured.

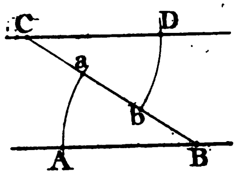
For (by N.B. Def. 8.) the length of the Lines or Sides makes no difference in the Angle; wherefore, if the Lines BC and BD were produced, equal to the Originals; i. e. equal 5 feet each; then would CD measure 6 feet 6 inches, and the Angle CBD would remain the same; which is obvious, if B and Bd be taken a half or a third part, or any other portion o BC or BD; for then, cd will be the same portion of CD; viz. a half or third, &c.

But, if ABC be the external Angle of a Building, so that, by reason of the obstruction of the Walls, &c. we cannot obtain the measure of the Chord Line, AC, it must then be got by its Complement of two Right Angles; i. e. by producing one Side, as AB, to D, and proceeding as before.

For, having assumed the Point B in the Right Line AD, and made the Angle CBD equal to the Complement of two Right Angles; the remaining Angle, ABC, is the inaccessible Angle required.

PROBLEM V. 31. I. Euclid.

To draw a Line through a given Point, which ſhall be parallel to a Right Line given.



AB is the given Line, and C is the Point given.

Through C, draw at pleaſure a Right Line, as CB, cuting the given Line, AB, in the point B.

With any Radius, on B, deſcribe the Ark Aa; and, on C, the Ark bD, with the ſame Radius.

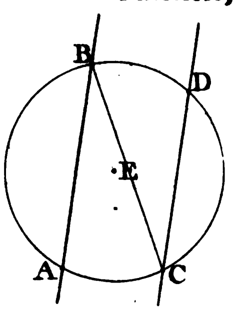
Make bD equal to Aa. - - - - Pr. 3.

and through the Points C and D, draw a Right Line.

Which will be parallel to the given Line AB. Q. E. F.

For, the Alternate Angles ABC and BCD are equal. Con.

Therefore, CD is parallel to AB. - - - P. 4. 1.



Otherwiſe, thus.

With any Radius, at diſcretion, ſet one Point of the Compaſſes in the given Point, C, and fix the other, at pleaſure, in E, either in the Line or out of it; on which Center deſcribe a Circle, paſſing through C; or draw the Arks AC and BD only.

Make BD equal AC, and draw CD;

Then is CD parallel to the given Line AB. Q. E. F.

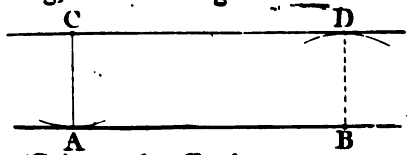
For, let CB be drawn.

DEM. Then, the Angles ABC, BCD are equal. Cor. 10. 3.

Therefore AB is parallel to CD. - - - P. 4. 1

Take

Or, when the Lines are very long, the following method is the moſt eligible.



Take the diſtance of the given Line AB, from the given Point C, in the Compaſſes; as CA; and, aſſuming any Point, B, towards the other end of the Line, deſcribe an Ark, at D, with that Radius.

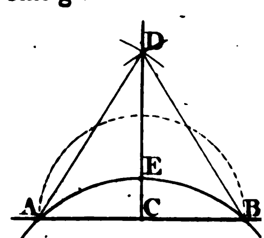
Apply a Ruler to the Point C and the Ark at D, and draw CD; which will be parallel to AB. - - Def. 7.

For, the Perpendiculars, AC and BD, being drawn, are equal, by Conſtruction.

PROBLEM VI. 11. I. Euclid.

To draw a Perpendicular, from any Point, in a given Right Line.

AB is the given Line, and, let C be the Point given.



With any Radius, on the given Point C, deſcribe a Semicircle, cuting the given Line in two Points, A and B; i.e. make CB equal CA.

Then, with any Radius, greater than AC, on A and B, deſcribe two Arks interſecting at D, and draw CD.

The Line CD will be perpendicular to AB. Q. E. F.

For, draw DA and DB.

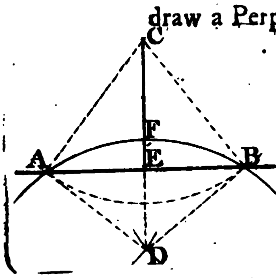
Dem. Then, the Triangles ADC, CDB, are congruous. Wherefore, the Angles are reſpectively equal, and ACD is equal DCB. - - - - - - P. 7. 1.
Therefore, CD is perpendicular to AB. - Def. 10.

After the ſame manner, a Perpendicular may be erected on the Ark of a Circle, AEB, at the point E.

PROBLEM VII. 12. I. Euclid.

To draw a Perpendicular to a Right Line, from a given Point which is out of the Line.

Let C be the given Point, from which it is required to draw a Perpendicular to AB.



On C, the Point given, defcribe an Ark, with any Radius greater than CE, the neareft diftance to the given Line, cuting it in A and B. - - - - - - - Poft. 4.
With the fame or any other Radius, on A and B, defcribe two Arks, interfecting at D. Apply a Ruler to the Points C and D, and draw CE; Then will CE be perpendicular to AB. Q. E. F.

For, draw AC and AD, BC and BD.

DEM. The Triangles ACD, DCB are congruous. Con.
Wherefore, the Angle ACE is equal to ECB.— P. 7. 1.
Conf. the Triangles ACE, ECB are alfo equal.— 8. 1.
And the Angles AEC, CEB, being equal, are confequently Right Angles. - - - - C. 2. 1. 1.
Therefore, CE is perpendicular to AB. - - Def. 10.

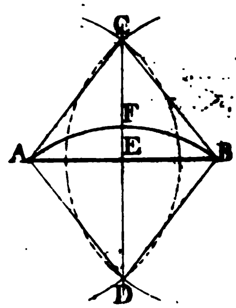
In the fame manner, a Perpendicular may be drawn to the Ark of a Circle, AFB.

PRO-

PROBLEM VIII. 10. I. Euclid,

To biſect a given Right Line, AB.

With any Radius, at pleaſure (provided it be more than half the given Line) on each extreme, A and B, deſcribe two Arks, cuting each other, at C and D.



Draw the Right Line CD; which will biſect, or divide into two equal Parts, the given Line AB, in the Point of their common interſection E. Q. E. F.

Draw AC and AD, BC and BD.

Dem. Then, ACBD is a Parallelogram; by Con. P. 15. 1. And, the two Diameters AB and CD biſect each other, in its Center, E. - - - - - P. 16. 1.

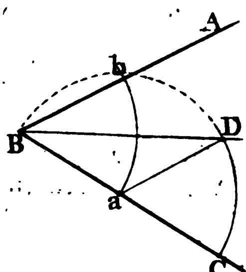
This Problem may be deduced from and proved as the foregoing.

After the ſame manner an Ark of a Circle may be biſected; as AFB in the Point F.

N. B. It is not neceſſary to draw the whole Arks from C to D, but only the Interſections at C and D; the reſt are uſeleſs.

PROBLEM IX. 9. I. Euclid.

To biſect a right-lined Plane Angle. ABC.



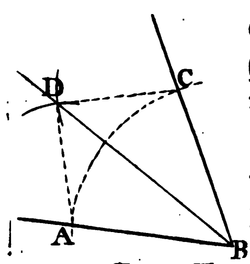
On the Vertex, B, of the given Angle, deſcribe an Ark, ab, at diſcretion, cuting both Sides, in a and b; and, on the interſection a, with the ſame Radius, draw the Ark CD. Make CD equal ab (Pr. 4.) and draw BD. Then will BD biſect the given Angle ABC.

Compleat the Semicircle CDAB, and draw aD.

Dem. Then DaC is an Angle at the Center, and DBC is at the Circumference of that Circle.

Th. DaC, (eq. ABC, Con.) is equal twice DBC. P. 9. 3.
And conſequently, ABC is double of DBC. Q. E. D.

Or, an Angle may be readily biſected in this manner.



On the Vertex B deſcribe an Ark, AC, as before. On the two Interſections, A and C, with the ſame, or any other Radius, deſcribe two Arks, cuting each other, at D.
Draw BD; which biſects the Angle ABC.

Draw DA and DC.

Dem. Then, the Triangles ADB and DBC are congruous, by Conſtruction.
Therefore, the Angle ABD is equal DBC. - P. 7. 1.

Angles may be thus divided into four, eight, or ſixteen equal Parts, by biſecting again and again; but there is no geometrical method, by which, Angles, or curved Lines, may be divided into any equal Parts, at pleaſure, as a Right Line may be divided; otherwiſe than by dividing the Ark with Compaſſes.

APPL.

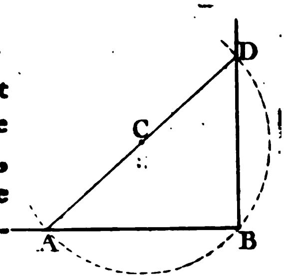
APPL. By this Problem, Carpenters and Joiners, &c. find, what they call, the Mitre of any Angle (whether it be Right, Acute or Obtuse) with ease and expedition.

In returning, or breaking Mouldings at an Angle, whether external or internal, the Angle bisected is the Mitre; in which the Mouldings will exactly fit each other.

PROBLEM X.

To make a Right Angle; or, to draw a Perpendicular at the extreme Point of a Right Line.

It is required to make a Right Angle, at the extreme, B, of the Right Line AB.



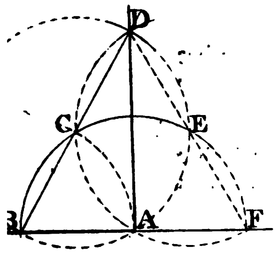
Set one point of the Compasses in the point B; and, with any Radius, at discretion, fix the other Point, at pleasure, at C (on that side, you require the Perpendicular) and draw the Ark ABD, through the Point B, and intersecting the Line AB in A.
Draw a Right Line through the Intersection A and the Center C, cuting the Ark on the opposite Side, at D. Draw BD, which will be perpendicular to AB. Q. E. F.

DEM. For the Angle ABD, being in a Semicircle, is a Right Angle. - - - - - P. 12. 3.

Or, it is frequently, and not inelegantly, performed thus.

It is required to draw a Perpendicular, to AB, at the extreme Point A.

Fix



Fix one point of the Compaſſes at A, and with any Radius, deſcribe the Ark BCE, cuting AB in B.

On B, deſcribe the Ark AC, cuting the former in C; on which Center deſcribe a Circle, or an Ark AED only; and on E, where it cuts the Ark BCE, draw ACD, or cut the other Ark, only, at D.

Draw AD, which will be perpendicular to AB. Q. E. F.

Or thus, without the Point E.

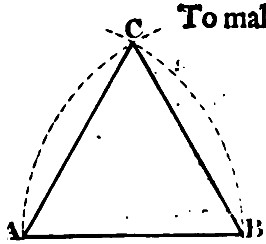
Deſcribe the Arks BC and AC, interſecting at C.
Through B and C draw the Right Line BCD, indefinite.
Make CD equal CB, and draw AD; and it is done.

By producing BA to F (making AF equal AB) and FD being drawn, it is demonſtrated as the 6th.

The uſe of Right Angles, perpendicular and parallel Right Lines, are ſo well known, that it would be impertinent to point them out; particularly to thoſe concerned in building, and various other mechanic Arts. They give beauty, ſtrength, utility, and conveniency to a Building and its ſeveral Appurtenances; alſo the execution, of the ſeveral parts thereof, depends on them entirely.

PROBLEM XI. 1. I. Euclid.

To make an Equilateral Triangle, on a Line given, AB.



With the Radius AB, the given Line, and on the extreme Points A and B, deſcribe two Arks AC and BC, interſecting at C.

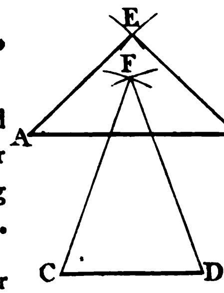
Draw AC and BC; and it is done. Q. E. F.

This needs no Demonſtration; for it is evident, that the three Sides are all equal, ſeeing, they are all Radii of equal Circles.

PRO-

PROBLEM XII.

To make an Isosceles Triangle.



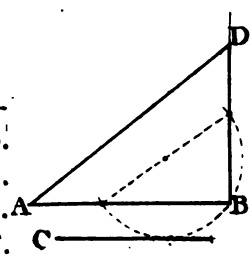
AB and C D are two given Lines; of which, let AB be the Base.

Take the Line CD in your Compasses, and with that Radius, on the extremes of the other Line, A and B, describe two Arks, intersecting at E; draw AE and EB; and it is done. Q.E.F.

If the other Line, CD, had been required for the Base, the operation would be the same; taking AB for Radius, & drawing the Arks, intersecting F; draw CF & FD.

PROBLEM XIII.

To make a Right-angled Triangle, having two Lines given for the Sides containing the Right Angle.



Let AB and C be the two given Lines.

On either Line, as AB, and, on either extreme, as B, make a Right Angle ABD.—10. Make BD equal to the other given Line, C,-2. and draw the Hypothenuse AD; - - Post. 2. ABD is the Triangle required. Q. E. F.

The use and application of Triangles, in general, are almost universal, in mathematical Sciences. The Isosceles and Right-angled are particularly useful in Perspective.

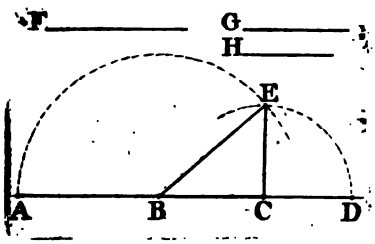
N. B. The Properties of the Right-angled Triangle are fully explained in the seventh Proposition of the sixth Book of Elements.

PRO-

PROBLEM XIV. 22. I. Euclid.

To make a Triangle, of three unequal Lines given; any two of which muſt be greater than the other.

F, G, and H, are the three given Lines.



Let them be placed at the ends of each other in one Right Line, in what order you pleaſe, i. e. having drawn the Right Line A D indefinite, make the Segments, AB, BC, and CD, reſpectively equal to the given Lines, F, G, H.

With the Radius AB, on B, deſcribe a Circle, or the Ark AE only; and on C, deſcribe the Ark DE, with the Radius CD, cuting the other Ark in E; draw BE and EC, and it is done. Q. E. F.

This Problem and the two laſt need no Demonſtration; every thing being as required, by Conſtruction; agreeable to Def. 26. 28. & 30.

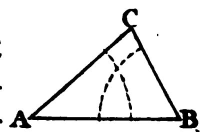
N. B. The Triangle BEC would be the ſame, if either of the other two Lines; F or A, had been made the Baſe, by placing it in the middle; only the poſition of the Triangle would be varied. This follows from Prop. 7. 1. of Elements.

2. There is no neceſſity for placing them, at all, in this manner; only, with the different Radii of two of the Lines, draw Arks, on the extremes of the other Line, cuting each other; taking any one for the Baſe.

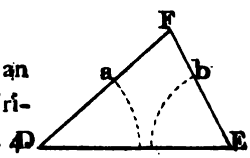
PRO-

PROBLEM XV.

To make a Triangle, fimilar to another Triangle*.



Let ABC be the given Triangle, and DE a Line given, on which to conftruct a Triangle; fimilar, and in a fimilar Pofition, i. e. a-like fituated, to ABC.



At the Point D, of the given Line, make an Angle EDa, equal to CAB, of the given Triangle, by - - - - - - Pr. 4.
And, at the Point E, make the Angle DEb, equal to ABC.
Produce Da and Eb, meeting in F. - - Poft. 3.
The Triangle DEF will be fimilar to ABC. Q. E. F.

DEM. For, the three Angles are refpectively equal. C. 5. 10. 1.
Therefore, the Sides are proportional. - - P. 4. 6.
Confequently, the Triangles are fimilar. - - Def. 1. 6.

APPL. This Problem is of great ufe in various Profeffions. For, by it we learn to take Altitudes and Diftances, though ever fo inacceffible; or, the Surveyor takes his Bearings, and lays down a Plan of the Ground he furveys; by it, the Mariner plans the Courfe in which the Ship ploughs the Ocean; and the Mechanic plans the Ground, on which he intends to build, &c.; in fhort, it is almoft of univerfal Ufe; which, to enumerate, is not neceffary in this place.

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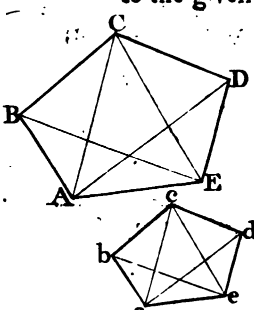
* For fimilar Figures, fee Def. 1. of the 6th Book of Elements.

PROBLEM XVI. 18. VI. Euclid.

To make a Figure, ſimilar to any given right-lined Figure.

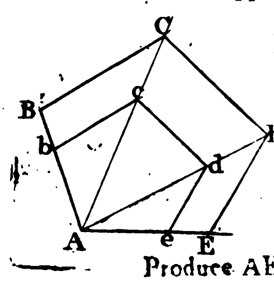
Let ABCDE be the Figure given, and ae a Line given, on which to conſtruct a Figure, ſimilar, and alike ſituated to the given Figure.



Draw the Diagonals AC, AD, BE, and CE.

On the given Line, ae, make the Triangle abe, ſimilar to the Triangle ABE, in the original Figure, by the foregoing. - - Pr. 15. alſo, make ace ſimilar to ACE, & ade to ADE Join bc & cd, which compleats the Figure. abcde is ſimilar to ABCDE. Q. E. F.

This method is deduced from the foregoing Problem; for, all right-lined Figures are compoſed of, or may be reduced into right-lined Triangles. It is demonſtrable from 4. and 13. of 6. El. The Application of this Problem is evident.



It may be thus performed, when required bigger.

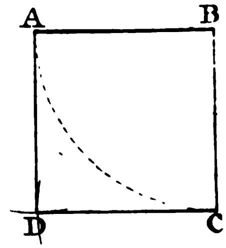
On AB, the given Line, deſcribe a Pentagon, Abcde, congruous to the given one, by reducing the given Figure into Triangles (as above) and making Abc, Acd, and Ade, reſpective ly equal to them. - - - - - Pr. 14.
Produce AE, and the Diagonals, Ac, Ad, indefinite.
Draw BC, CD, and DE, parallel to the Sides bc, cd, and de, reſpectively; cuting the Diagonals and Side AE, in the Points C, D, and E. ABCDE is ſimilar to Abcde.

After the ſame manner, any Poligon may be increaſed or diminiſhed in any Proportion; demonſtrable by 2. 6. El.

PRO-

PROBLEM XVII. 46. I. Euclid.

To make a Square on a given Line, AB.

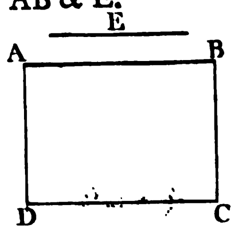


On either extreme of AB, make a Right Angle; as A B C. - - - - Pr. 10.
Make B C equal A B. - - - Pr. 2.
Then, with the Radius A B, on A and C, describe two Arks, interfecting at D, and draw AD and DC.
The Quadrilateral ABCD is a Square.

DEM. For it is equilateral, by Conftruction;
confequently it is a Parallelogram. - - - P. 15. 1.
Wh. the Angle D = B; conf. D is a Right Angle.-10. 1.
But, the Angles A and C are alfo equal; conf. Right.
Therefore, A B C D is a Square. - - - Def. 35.

PROBLEM XVIII.

To make a Rectangle; two Lines is given; AB & E.



Make ABC a Right Angle. - - - Pr. 10.
Make B C equal to the Line E; - - - 2.
Draw AD parallel to BC, and CD to AB.--5.
ABCD is the Rectangle required.

DEM. For, it is a Parallelogram, - - Con.
Wh. the Angle D is equal B, and A equal C. - P. 15. 1.
But, the Angle B is a Right one. - - - Con.
wherefore they are all Right Angles; - - Th. 1. 10. 1.
and therefore it is a Rectangle. - - - Def. 34.

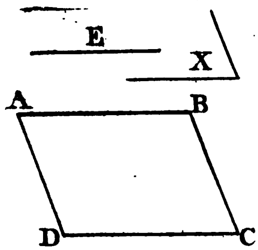
The Square and Rectangle are of great ufe in the mechanic Arts; as moft regular Figures are Rectangles.
In Menfuration, the Area of every Figure is reduced to a ftandard meafure; by the Square, or other Rectangle.

P R O-

PROBLEM XIX.

To conftruct a Parallelogram under a given Angle, and having its Sides equal to given Lines, of determinate length.

AB and E are the given Lines, and X the given Angle.



At either extreme, of either Line (as A) make an Angle, BAD equal to the given Angle. Make AD equal to E, the other given Line. Through the Points D and B, draw DC parallel to AB, and BC to AD, meeting in C.-Pr. 5. Then is ABCD the Parallelogram required.

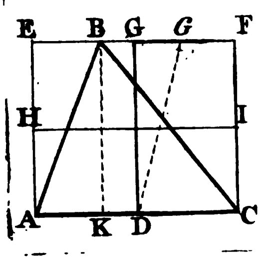
This needs no Demonftration, every thing being as required by Conftruction.

The oppofite Sides and Angles are equal, by - - P. 15. 1.

APPL. By this Problem is delineated Plans, &c. of four fided Objects which are not right-angled, but, whofe oppofite Sides are equal. Any one Angle, being taken, determines all the reft.

PROBLEM XX. 42.I.Euclid.

To make a Rectangle, or any other angled Parallelogram, equal to a given Triangle.



ABC is the given Triangle.

Bifect any Side, as AC in the Point D.—8.

Draw EF, through the oppofite Angle, B, parallel to AC. - - - - Pr. 5.

Draw DG, from the point of bifection, perpendicular to the Bafe AC, cuting EF in G.

Laftly, draw CF parallel to DG.

The Rect. DGFC is equal to the Triangle ABC. Q. E F.

Or,

Or, if the Rectangle AHIC be constructed on the whole Base, AC, and half the perpendicular Altitude, BK, it will be equal to the Triangle ABC.

DEM. For, it is half the Rectangle AEFC, which is double the Area of the Triangle ABC. - - 17. 1.

N. B. If any other angled Parallelogram was required, you must proceed as in Prob. 19. making the Angle CDG or DCF equal to the Angle given; still keeping the same Altitude; i. e. between the same Parallels, AC and EF, and the same Base DC or AD.

SCHOL. *From hence, and from the 17th Proposition of the first Book of Elements, the whole Theory of Mensuration is deduced; as all Figures whatever, except Parallelograms, are resolved into Triangles in Mensuration.*

From the 17th of the first Book we learn, that every Triangle is equal to half a Parallelogram of the same Base and Height; consequently, the Rectangle DGFC *is equal to the Triangle* ABC, *which is on half its Base,* AC, *and the same height,* GD, *equal* BK.

COR. Hence, the Rule for measuring a Triangle is to multiply the Perpendicular, BK (equal GD) by half the Base, AC (equal DC) or the whole Base, AC, by half the Perpendicular BK (equal AH); the first gives the Rectangle DGFC, the other is the Rectangle AHIC; either of which is equal to the Triangle ABC.

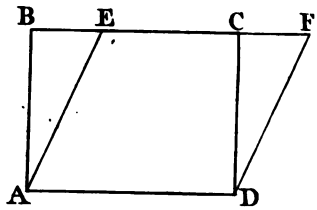
For, if the whole Base AC be multiplied by the Perpendicular BK, it gives the Area of the Rectangle AEFC which is double of the Triangle (17. 1,) consequently, half that Sum is the Area of the Triangle ABC; and also of the Rectangle DGFC or AHIC; which is, therefore, equal to the Triangle given.

PRO-

PROBLEM XXI.

To make a Parallelogram, under any given Angle, equal to a Rectangle, and having the same Base.

ABCD is the given Rectangle.



Make the Angle DAE equal to the given Angle, cuting BC in E, by - - - Pr. 4.
Produce BC; and draw DF parallel to AE.-5.
The Parallelogram AEFD is equal to the given Rectangle ABCD. - - - - P. 18. 1.
Or, it may be thus demonstrated.

DEM. AB is equal DC (15. 1.) and BE is equal CF. Ax. 7. for, EF (eq. AD) is equal to BC; and EC is common. And, the Angle ABE is equal to DCF. - - P. 4. 1. conf. the Triangle AEB is equal to DFC. - - 8. 1. Wherefore, if from the Rectangle ABCD there be taken away the Triangle AEB, and an equal Triangle DFC be added, the Par. AEFD is equal to the Rect. ABCD.

COR. 1. Hence it is evident, that two Spaces may have the same Area, yet differ greatly in compass or circuit.

For, the Sides AE and DF, of the Parallelogram AEFD, are greater than AB and DC of the Rectangle ABCD. 12. 1. But, EF is equal BC (Ax. 3.) for each is equal AD.—15. 1. Therefore, the Circuit of the Par. AEFD is greater than that of ABCD; and they contain equal Areas. - - 18. 1.

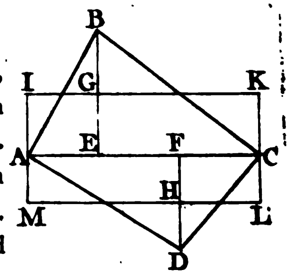
COR. 2. From hence is deduced the general Rule for measuring all Parallelograms; which is, to multiply the Base; i. e. any Side (as AD) by its perpendicular height, (CD)

For it gives the Area of the Rectangle ABCD, and consequently of the Par. AEFD, which is equal to the Rectangle.

PRO-

PROBLEM XXII.

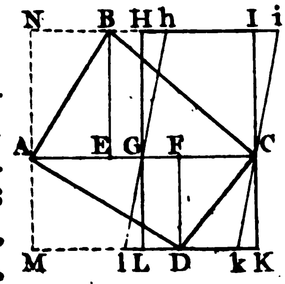
To make a Rectangle equal to a given Trapezium; ABCD.



Draw either Diagonal, as AC; to which, draw the Perpendiculars BE and DF, from the oppoſite Angles. - - - Pr. 7.
Biſect the Perpendiculars, BE and FD, in G and H. - - - - - Pr. 8.
Through the Points G and H, draw IK and LM parallel to AC. - - - Pr. 5.
and through A and C, draw IM and KL, perpendicular to AC. - - - Pr. 10.
The Rectangle IKLM is equal to the Trapezium ABCD.

DEM. For, the Rect. AIKC is equal to the Triangle ABC, and the Rect. ACLM is equal to the Triangle ACD. 20.
Conſ. the Rect. IKLM is equal to the Trap. ABCD. Ax. 2.

Otherwiſe,



Having drawn a Diagonal and Perpendiculars, as before; biſect the Diagonal, in the Point G. - - - - Pr. 8.
Draw BI and MK through the Angles B and D, parallel to the Diagonal (AC), and, through G and C, draw HL and IK, parallel to BE and FD, cuting BI and MK in the Points H and L, I and K.
Then, the Rect. HIKL is equal to the Trap. ABCD.

DEM. For, the Rect. GHIC is equal to the Triangle ABC. and the Rect. GLKC is equal to the Triangle ACD. 20.

N. B. By this Problem and the laſt may be performed the 45th of Euclid, without the aſſiſtance of the next, which he makes uſe of, and much readier; making the Parallelogram under a given Angle, as h i k l, which is equal to HIKL by the laſt.

Cor. Hence we learn, to find the Area of any Quadrilateral whatever; the Rule for which is, to multiply the Diagonal (AC) by half the Sum of the two Perpendiculars (Fig. 1.) or, the Sum of the two Perpendiculars (BE and FD) by half the Diagonal (GC, Fig. 2.) or, if we multiply the whole Diagonal by the Sum of the two Perpendiculars, they will give the Area of the Rectangle MNIK; half the Sum, of which, is the Area of the Trapezium ABCD.

It may appear ſtrange, to thoſe who have not conſidered it, that the Area of any Figure ſhould be obtained without meaſuring its Sides; which are of no uſe in this operation.

But, having well digeſted what has been advanced, they will find, that the whole buſineſs of Menſuration is to find a Rectangle equal to any Figure; for (as I ſhall make appear hereafter) the multiplication of any two Numbers, applied to meaſure, denotes a Rectangle of ſuch Dimenſions.

Now, ſince it ſeldom happens, that a Trapezium has either Right Angles or parallel Sides, it is plain, that they cannot be of any uſe towards obtaining its Area; whereas, the Diagonals and Perpendiculars are at Right Angles with each other.

A Diagonal divides any Quadrilateral into two Triangles; and every Triangle is equal to half a Parallelogram having the ſame or an equal Baſe, and the ſame Altitude. (Prop. 17. 1.)

Hence, it is eaſy to account for the Rules given for meaſuring a Trapezium, as two Triangles having a common Baſe; which is a Diagonal of the Trapezium.

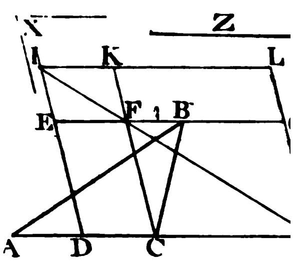
For, every Trapezium is equal to half a Parallelogram which circumſcribes it, having two Sides parallel to either Diagonal; and all Parallelograms having the ſame Baſe and Altitude are equal (18. 1.) conſequently, they are equal to a Rectangle of thoſe Dimenſions.

PRO-

PROBLEM XXIII. 44. I. Euclid.

To make a Parallelogram, equal to a Triangle, having an Angle equal to a given one, and a Side equal to a given Right Line.

ABC is the given Triangle; X the Angle, and Z the given Line.



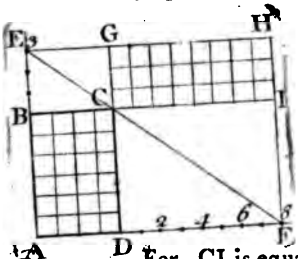
Make the Parallelogram DEFC, equal to the Triangle ABC, whose Angle, at D or C, is equal to the given Angle, X.—Pr. 20. Produce DE indefinite; also, produce AC; and make CH equal to the given Line, Z;-2. From the Point H, and through the Angle F, of the Parallelogram DEFC, draw HI, cuting DE, produced, in I.
EI, is the other Side of the Parallelogram, sought; which may be compleated by Prob. 19.

Draw IL parallel to AH, and HL parallel to DI meeting at L; produce EF to G, and CF to K.

DEM. The Par. KLGF (having its Angles, at K and G, equal to the given Angle X, and a Side, FG, equal to a given Line, Z) is equal to the Par. DEFC.—P. 19. 1. which, is equal to the Triangle ABC. - - Pr. 10.

SCHOL. *This is, properly, geometrical Division.*

Let the Rectangle ABCD be any Quantity given, to be divided; whose Area is 24; the Side AB being 6, and AD 4 equal Parts; it is required to be divided by 3.



Produce any Side, as AB; and make BE equal 3 (half AB) also produce AD indefinite. Through E and the Angle C, draw EC, and produce it, cuting AD produced, in F; DF will be equal 8 of the divisions, or twice AD, which is the Quotient sought.

Compleat the Rect. AEHF; produce BC to I, and DC to G; the Rect. CH = AC.

For, CI is equal DF, i. e. equal 8; and, CG is equal BE equal 3; 8 multiplied by 3, is equal 24, the Area of CH; which is equal to the Dividend, or the Rectangle ABCD.

If the Divisor had been 8, equal DF, the Quotient would have been 3, equal BE.

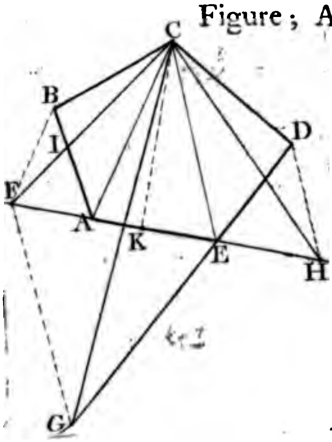
If a fractional Number was given for the Divisor, the Quotient would, most probably, be fractional likewise; for the Rectangle under the Divisor and the Quotient will always be equal to the given Rectangle, which is the Dividend.

As in Division, the Quotient multiplied by the Divisor is (when there is no remainder) equal to the Dividend; which proves the work to be true.

N. B. It is equal, which way the given Rectangle is situated, in this operation; or which Side is produced for the Divisor; the Quotient, will always be the same.

PROBLEM XXIV.

To make a Triangle equal to any given Right-lined Figure; ABCDE.



Draw the Diagonals AC and CE; and produce EA, indefinite. - - - Post. 3.

From the Angle B, and parallel to AC (the adjacent Diagonal) draw BF; cuting EA, produced in F; and draw FC.

Again. Produce the adjoining Side DE, indefinite; and, parallel to the other Diagonal CE, draw FG; cuting DE produced, in G; draw GC, and it is done. Q. E. F.

The

The Triangle GCD, is equal to the given Pentagon ABCDE.

DEM. 1ſt; the Trap. FCDE is equal to the Pent. ABCDE. For, ſince FB is parallel to AC, the Triangle AFC is equal to ABC (18. 1.) and AIC is common to both; wherefore, AFI is equal to IBC. - - Ax. 7. 1.

Then, if from the given Pentagon, ABCDE, there be taken away the Triangle IBC, and its equal FAI be added; the Trapezium FCDE is equal to the given Pentagon. - - - - - - Ax. 7. & 6.

But, the Trap. FCDE is equal to the Triangle GCD. For, becauſe FG is parallel to CE, the Triangle EGC is equal to EFC. - - - - - - - P. 18. 1.
conſ. EGC added to ECD = EFC + ECD.—Ax. 6. 1.
i. e. the Triangle GCD is equal to the Trap. FCDE.
But, the Trap. FCDE = the Pent. ABCDE.-proved above
Th. the Triangle GCD is equal to the given Pentagon.

Or, if the Side AE had been produced both ways, and DH drawn parallel to CE, cuting AE in H, and if HC be drawn; the Triangle FCH is alſo equal to the given Pentagon, equal to the Triangle GCD.

For, the Triangle CHE is equal to CDE. - - 18. 1.

SCHOL. *By which means, a Triangle may be readily conſtructed, whoſe Altitude ſhall be equal to the Altitude of the Poligon; or, to a Perpendicular, from any Angle to any Side; as* CI *or* CK. *Or, any one Angle, as* D, *and an adjoining Side,* CD, *of the Poligon, may be retained in the Triangle.*

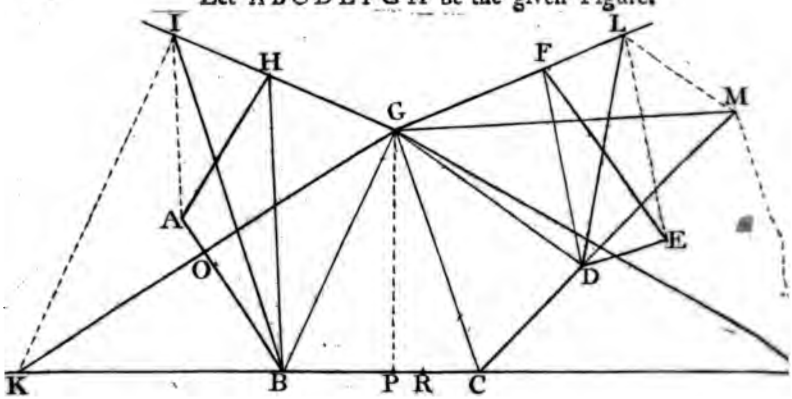
I have been more particular in the Demonſtration of this Problem than in any of the former; becauſe, it may, to ſome, appear, at firſt ſight, rather intricate; which it certainly is not, being properly analized.

From hence, and from Prob. 20. a Rectangle may be found or conſtructed equal to any given right-lined Figure; and, by Prob. 25. the Side of a Square, equal to a given Rectangle, is readily obtained.

But

But, as the operation, in a more complex Figure, may not be readily performed, without an Example, I ſhall briefly go through the whole proceſs; and alſo, ſhew how much it is preferable to the 14th Prop. of the 2d Book of Euclid; viz. To find the Side of a Square, which is equal to any given right-lined Figure.

Let A B C D E F G H be the given Figure.



Firſt, draw the Diagonals B H, B G, G C, &c.
Produce the Side G H, and, draw AI, parallel to the Diagonal B H; cuting G H produced, at I, and draw B I.

Now, ſince AI is par. to BH, the Tri. BIH,=BAH. 20. 1.
wherefore, the Tri. BIG, is equal to the Trap. BAHG.

Next, produce the Side BC, both ways, indefinite, for the Baſe of the Triangle ſought.

From the new-acquired Angle I, draw I K, par. to the Diagonal BG; cuting CB produced, at K, and draw GK. Then, the Tri. BKG=BIG, having the ſame Baſe BG. 18. 1.

But the Tri. BIG is equal to the Trap. BAHG;
wh. the Tri. BKG, is equal to the Trap. BAHG.---Ax. 3.
Conſ. the Triangle BKO, which is added, is equal to the Trap. A H G O, taken away from the original Figure; for, B O G is common to both.

Again,

Again; produce GF; and, from the Angle E, draw EL, parallel to the Diagonal FD; and draw DL.
The Triangle FLD is equal to FED. - - - 18. 1.
The given Figure is now reduced to a Pentagon, KGLDC.

Then; produce the Side CD; and, from the Angle L, draw LM, parallel to the Diagonal GD; cuting CD produced, in M, and draw GM.
The Triangle GMD is equal to GLD. - - 18. 1.

Laſtly; from the Angle M, and par. to the Diag. GC, draw MN, cuting KC produced, in N, and draw GN.
The Triangle GNC is equal GMC; equal to the Trapezium GLDC; equal GFEDC.

But, the Triangle BKG is equal the Trap. BAHG; wherefore, BKG, added to BGC, added to GNC, is equal to the Triangle KGN. - - - Ax. 2.
Conſ. the Triangle KGN, is equal to the Trap. KGMC; which is equal to the Pentagon KGLDC;
equal to the Hexagon KGFEDC;
equal to the Heptagon BIGFEDC;
equal to the Octagon AHGFEDCB.
Therefore, the Triangle KGN, is equal to the given Figure.

Having thus reduced the given Figure to a Triangle, it is eaſily formed into a Rectangle, by Prob. 20.;
For, a Rectangle on KR, half the Baſe KN, and GP, the perpendicular Altitude of the Triangle, is equal to the Triangle. - - - - Cor. 17. 1.

Or, by Prob. 23. it is eaſily reduced to a Parallelogram under any given Angle and Side.

SCHOL. Let

SCHOL. *Let any one compare this, with the 14th Propoſition in the 2d Book of Euclid; that is, let him go through the operation, both ways, in a Figure of as many Sides; I am confident to which he will give the preference. To ſay nothing of the inaccuracy of the other, this may be done in leſs than a fourth part of the Time. Each Triangle (of which there are ſix, in this Figure) goes through two operations, viz.* 20*th and* 23*d of this; and are added, ſeparately into one Sum, or Rectangle. Whereas, by this method, the Figure is reduced one Side at every operation; from a Hexagon to a Pentagon, from a Pentagon to a Trapezium, from a Trapezium to a Triangle; each, being equal to the original Figure: and, from a Triangle to a Rectangle, or other Parallelogram under any Side or Angle; by the* 23*d.*

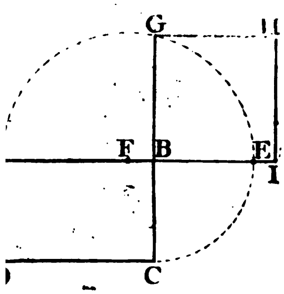
APPL. This Problem, may be applicable in ſeveral Caſes, in ſurveying, &c. with the greateſt accuracy. e. g.

If you would exchange an irregular piece of Ground in one Place, for the ſame Quantity in another, ſuppoſe to build on; which, by reaſon of the contiguous Buildings, is confined to a certain Angle, which is the given Angle; the length in Front may be conſidered as the given Side; the Queſtion is, what Depth of Ground from the Front is required, to be equal in its Area to the other.

Having firſt reduced the original given Figure to a Triangle, by this Problem, it is then convertible into a Parallelogram under any Side and Angle, by the 23d; or, into a Square by the following.

PROBLEM XXV.

To make a Square, equal to a Rectangle. ABCD.



Produce any Side, as A B, of the given Rectangle, until it be equal to the adjoining Side, BC; i. e. make AE equal to AB and B C, in a Right Line.

Biſect AE, in F; on which, with the Radius AF, equal FE, deſcribe a Semicircle.

Produce CB, till it cuts the Arch, at G; or, at the point B, draw BG perpen. to AE.

Deſcribe the Square BGHI on the Line BG.

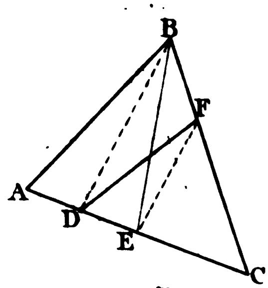
DEM. It

DEM. It will be equal to the Rectangle ABCD. - - 14. 3.
Alſo, BG is a mean Proportional between AB and BE.
Conſ. the Rectangle under AB and BE (i. e. DB) is equal to the ſquare of B G (i. e. BH) by—Cor. to 9. 6.

N. B. In the performance of this Problem it is not neceſſary to conſtruct the Rectangle; but only, to draw a Right Line (AE) in which, take AB and BE equal to the meaſures of the two Sides, and proceed as above.

PROBLEM XXVI.

To divide a Triangle into two equal parts, by a Right Line drawn from any Point given, in a Side.



Let D be the given Point, in the Side AC, of the Triangle ABC.

Biſect the Side AC, in E, and draw EB, to the Vertex B.
Draw DB, and EF parallel to it, cuting B C in F, and draw DF.
The Trapezium ABFD, is equal to the Triangle DFC.

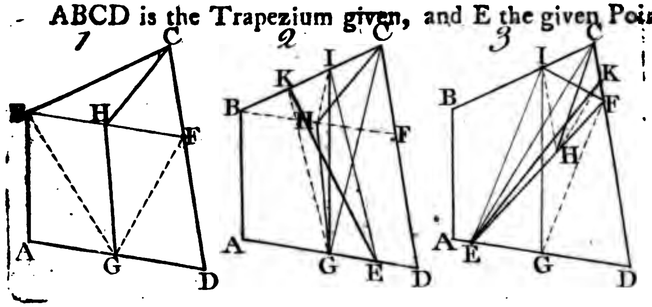
DEM. Becauſe, AE=EC, the Tri. ABE=EBC.— 20. 1.
But, EF is par. to DB; wh. the Tri. BFD=DBE--ſame.
conſ. BFD+ABD is equal to DBE+ABD. - - Ax. 6.
And, the Triangle ABE is equal to EBC. - - - above.
wh. the Trap. ABFD, is equal to the Tri. ABE, eq. EBC; and conſequently, to the Triangle DFC.

Therefore, the Right Line DF divides the Triangle ABC into two equal Parts. Q. E. F.

PROBLEM XXVII.

To divide a Trapezium into two equal Parts, by a Right Line, drawn from a Point given in any Side.

ABCD is the Trapezium given, and E the given Point.



Through the Angle B, draw BF, parallel to AD. — 5. Bisect AD and BF, in G and H; and draw GH and HC. Then the Pentagon ABCHG is equal to the Trap. AHCD.

DEM. For (having drawn BG and GF) the Triangle ABG is equal to GFD, the Triangle BGH is equal to HGF, and BCH is equal to HCF. - - - - P. 18. 1.

2dly. In Fig. 2. let ABCHG be equal GHCD, as before. Draw GC, and HI parallel to GC, cuting BC in I, and draw GI.
The Right Line GI bisects the given Trapezium.

DEM. For, because HI is parallel to GC, the Triangle GIC is equal to GHC. - - - - - P. 18. 1.
Wherefore, GIC added to GCD=GHC+GCD. Ax. 6.

Again. Draw EI, and GK parallel to EI, cuting BC in K, and draw EK.
I say, that EK divides the Trapezium ABCD equally.

DEM.

DEM. For firſt, the Trap. GHCD was proved half ABCD; ſecondly; GICD was proved equal to GHCD; and, EKCD is equal to GICD.
For the Triangle EKI is equal to EGI. - - P. 18. 1.
Therefore, the Trap. ABKE is equal to EKCD.

Now, if E had been given near the Angle A, at the leſſer end of the Trapezium, the operation would require more labour.

Let Fig. 3. be ſuppoſed the ſame Figure divided, by the Line GI, into two equal parts, from the middle Point G (as by the ſecond operation) which is neceſſary to be firſt done, in all caſes.

Join the given Point, E, and I, as before; draw GF, parallel to EI, cuting CD in F, and draw EF and IF. Then, the Pent. ABIFE is equal to the Tri. EFD + FIC.

For, becauſe GF is parallel to EI, the Tri. IFE = EGI.

But, although the Pentagon ABIFE is equal to half the Trap. ABCD, it is not equally divided by one Right Line. Therefore, draw IH parallel to CD, cuting EF in H, and draw HC.

Then, becauſe IH is parallel to CF, the Triangle ICH is equal to IFH. - - - - - - - 18. 1.
Wh. the Pent. ABCHE = ABIFE = the Trap. EHCD.

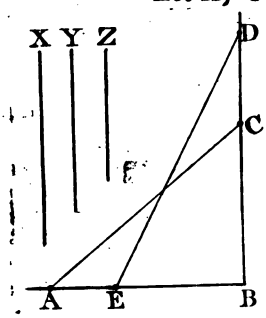
Laſtly; join EC, and draw HK parallel to EC, cuting CF in K; and draw EK; which Line divides the Trapezium ABCD into two equal parts, as required.

For, ſince HK is parallel to EC the Tri. EKC = EHC; wherefore, EKC added to ECBA = EHC added to ECBA. But, ABCHE was equal to ABIFE, equal to ABIG; which, was equal to half the Trapezium ABCD.

PROBLEM XXVIII.

To find the Side of a Square, equal to any Number of Squares.

Let X, Y and Z be the Sides of three given Squares.



It is required to find the proportion of a Line, on which if a Square be conſtructed, it ſhall be equal in Area to all the three.

Make a Right Angle ABD. - - Pr. 10.
From the Angle B, take BA and BC, equal to any two of the given Lines, reſpectively, as X and Y, and join AC.
The Square of AC is equal to the two Squares of AB and BC, or X and Y. - - - P. 20. 1.

Again. Make BD equal AC, and BE equal Z;
Draw ED, which is the Line required.

DEM. For, ED ſquare = the two Squares of EB & BD.
But, BD, equal AC, ſquare = AB + BC ſquare.
Wherefore, ED ſquare = AB + BC + EB ſquare.
i. e. ED ſquare = the three Squares, of X, Y and Z. - - 20. 1.

APPL. By this uſeful Problem, Quantities may be increaſed in any Proportion at pleaſure.
Alſo, by means of this Problem, and Prop. 20. 1. Carpenters form a Right Angle, in framing Timber, &c.
For, having made AB equal 3 feet, and BC equal four; then, if AC meaſures 5 feet ABC is a Right Angle.

N. B. The Numbers 3, 4 and 5 being multiplied, ſeparately, by any one Number, at pleaſure, produce the ſame effect. e. g.
If AB be made 6 or 9 feet, and BC equal 8 or 12 feet, then will AC be equal to 10 or 15 feet, if the Angle ABC be a right one.
For, the Square of 9 is 81; and the Square of 12 is 144, added to 81 is 225; equal to the Square of 15.———20. 1. El.
When

When the Timbers are long, it is neceſſary, in placing them at right angles, to apply greater meaſures; which, will give the Right Angle with greater accuracy.

Cor. Hence, a Perpendicular may be drawn, very readily, at the extremity of a Right Line; by a Scale of equal Parts. As B C, perpendicular to AB. e. g.

Take, by any Scale, 3, 6, 9, or 12 equal Parts, which, ſet off, from B to A; at which point, B, a Perpendicular to AB is required.

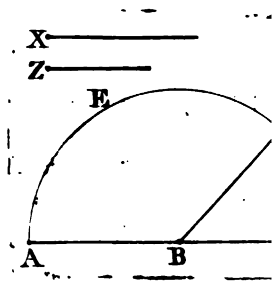
Then take, for Radius, 4, 8, 12, or 16 Diviſions, of the ſame Scale; and, ſeting one Point of the Compaſſes in B, make a ſmall Ark, at C.

By the ſame Scale, take 5, 10, 15, or 20 equal Parts, and with one point of the Compaſſes, at A, croſs the former Ark, at C, and draw B C; which will be perpendicular to AB.

For, the Square of A C is equal to the two Squares of A B and B C, added together; by 20. 1. El.

PROBLEM XXIX.

The Sides of two Squares being given, to find the Side of a Square which is equal to the difference between them; i. e. by how much the greater exceeds the leſs.



X and Z are the Sides of the given Squares.

Draw AC indefinite; in which, take AB equal X, and BC equal Z. - - - Pr. 3. Make ACD a R. Angle, and draw CD, indef. On B, with the Radius AB, deſcribe the Ark AED, cuting CD at D; CD is the Side of a Square, equal to the difference between the Squares of AB and BC, or X & Z. Draw BD.

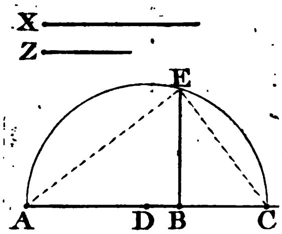
Dem. BD (equal AB, equal X) is the Hypothenuſe of the right-angled Triangle DCB; the ſquare of which, is equal to the ſquare of BC added to the ſquare of CD. Conſequently, BD ſquare (equal X) exceeds BC ſquare (equal Z) by the ſquare of CD; by Prop. 20. 1.

P R O-

PROBLEM XXX. 13. VI. Euclid.

To find a mean Proportional between two given Lines.

X and Z are the two given Lines.



It is required to find a Line, to which, either of the Lines, X or Z, ſhall have the ſame Ratio or Proportion, as that Line has to the other. Or, the Square of which, ſhall be equal to a Rectangle under the two given Lines.

Draw at pleaſure A C; make AB equal to one of the given Lines, as X, and make BC equal to the other, Z.
Biſect A C, in the point D; on which Center, and with the Radius A D, equal D C, deſcribe a Semicircle.
At the Point B, draw a Perpendicular, to A C, cuting the Ark at E; and B E is the Line ſought.

DEM. For, draw AE and EC; AEC is a Right Angle--12. 1.
And, the Perpendicular, BE, in a right-angled Triangle, AEC, is a mean Proportional, between the Segments of the Baſe, AB, BC, made by the Perpendicular. C. 1. 7. 6. El.
Conſequently, AB is to BE, as BE is to BC. — ſame.
Therefore, AB×BC = the ſquare of BE.—Cor. to 9. 6.

SCHOL. *This is the very ſame, in the operation, as the 25th; for the Side of a Square is a mean Proportional between the two Sides of a Rectangle having an equal Area.*

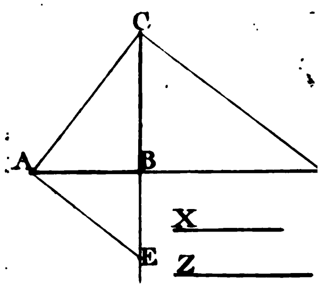
PRO-

PROBLEM XXXI. 11. VI. Euclid.

To find a third Proportional to two given Lines.

X and Z are the given Lines.

It is required to find a third Line, to which the greater, Z, ſhall have the ſame proportion, as the leſſer, X, has to Z. Or, which ſhall have that proportion to the leaſt, as the leaſt to the greateſt.



Make a Right-angled Triangle, ABC, whoſe Catheti or Legs, are equal to the two given Lines, i. e. make AB equal to one (X) and BC equal to the other.---13 Produce AB and CB, indefinite, Make ACD and CAE Right Angles; i. e. draw CD and AE perpendicular to AC, cuting AB and CB, produced, in D and E.

Then will BD be a greater, and BE a leſs, third Proportional, to the two given Lines, X and Z; Q. E. F.

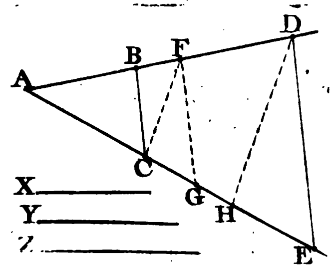
DEM. For, as in the laſt, EB is to AB, as AB to BC; and AB : BC : : BC : BD. - - - - P. 7. 6. Wherefore, EB, AB, BC, and BD are in continual Proportion. - - - - - - Def. 7. 5.

Conſequently EB is a leſs, and BD a greater third Proportional, to the two Lines AB and BC, equal X and Z.

PRO-

PROBLEM XXXII. 12.VI.Euclid.

To find a fourth Proportional, to three given, unequal Lines; X, Y and Z.



It is required to find a Line, to which, the third (Z) ſhall have the ſame proportion, as the firſt (X) has to the ſecond (Y). Or, the ſame proportion to the firſt Line as the ſecond has to the third.

Draw two Lines, AD & AE, making any Angle at pleaſure.
Make AB equal X, & AC equal Y; alſo make BD equal Z,
Draw BC, and DE parallel to it; cuting AE in E; and,
CE is a fourth Proportional, greater than Z.
i. e. As AB is to AC, ſo is BD to CE. Q. E. F.

Dem. For, becauſe BC is parallel to DE, the Sides of the Triangle ADE are cut proportionally;
wherefore, AB :: AC, as BD :: CE. - - P. 2. 6.
But, AB is equal to X, AC is equal Y, and BD equal Z;
therefore, as X is to Y, ſo is Z to CE.

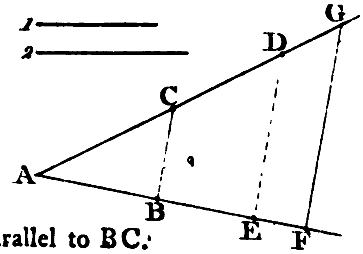
If a leſs Proportional be required; make AG equal to Z, make AF equal Y, and GE equal to X.
Join FG, and draw DE parallel to it.
Then, will FD be the Proportional ſought.

Dem. For, as AG (equal Z) is to AF (equal Y) ſo is GE (equal X) to FD. - - - - by the ſame.

After

After the ſame manner, a third Proportional may be found.

Draw AF and AG, making any Angle, as before.
Let AB & AC be made equal to the given Lines, reſpectively



If the greater Proportion be required, make B F equal to A C, and draw F G, parallel to B C.

The Segment CG is the Proportional ſought.

But if a leſs Proportional is wanted; make CD equal AB, and draw DE parallel to BC.
Then, BE is the Proportional required.

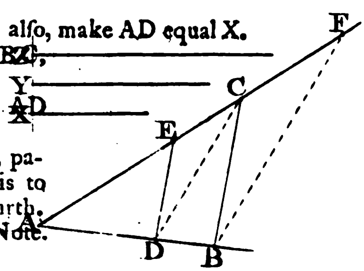
DEM. For, as AB is to AC, ſo is BF to CG;
and, as AC is to AB, ſo is CD to BE. - - 2. 6. El.
But, B F was made equal to A C, and CD to AB.
Conſ. as AB : A C :: AC: CG ; and, AC : CD :: CD : BE.

N. B. A fourth Proportional to three given Right Lines may be required and found in various orders of the given Lines.

Let X, Y, and Z be three given Lines; of which, let X be the leaſt, and Z the greateſt of the three,
By the firſt, it is, as X is to Y, ſo is Z to a fourth;
and by the ſecond, as Z is to Y, ſo is X to a fourth.
But it may be, as Y is to either X or Z, ſo is the other to a fourth.
Alſo, as X is to Z, or Z to X, ſo is Y to a fourth.

It may alſo be obſerved, that it is not neceſſary to draw the Lines, making an Angle, for the operation, longer than the greateſt given Line; except when a Proportional is required greater; for the meaſures may all be ſet off from the Vertex. e. g.
Let AB and AC make any Angle (BAC) at diſcretion.

Make AB equal to Y, and AC equal Z; alſo, make AD equal X.
Join BC; and draw DE, parallel to BC,
cuting AC in E.
Then, as AB or Y, is to AC or Z; ſo is AD or X, to AE, a fourth.
But, if DC be joined, and BF drawn, parallel to DC; it is then, as AD or X, is to AC or Z, ſo is AB or Y, to AF, a fourth.



K Note.

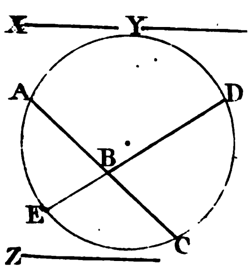
Note. This method, of ſeting off all the meaſures from the Angle, is the moſt eligible, when we are clear in the manner of placing them; being leſs liable to error than when they are ſet forward on a longer Line; on account of the Parallels being nearer together. The Demonſtration is in the 4th of 6th El.

A third Proportional may alſo be found after the ſame manner. For, if AD be the firſt; and, if AB and AC be each made equal to the ſecond; DC being joined, and BF drawn parallel to DC gives AF, a greater third Proportional; or, if AF be the firſt, AD is a leſſer third.

A third or a fourth Proportional may be very elegantly found after this manner.

Let X, Y and Z be three given Lines.

If you require a leſs Proportional, X is the firſt Term taken; if a greater be required, Z muſt be the firſt, contrary to the order, after the former method; nor can the Terms be taken alternately, as in the other.



Draw a Right Line, AC, at pleaſure. Make AB and BC equal, reſpectively, to the firſt and ſecond Terms, X and Y. Through the point B, draw, at pleaſure, DE and make BD equal to the third Term (Z) Deſcribe a Circle through the three Points A, C, and D, cuting DE, at E.—Pr. 40. Then, BE is a fourth Proportional; in the order Z to Y, as X to BE.

DEM. For, the Rectangles under the ſegments of Chord Lines, cuting each other, are equal, i. e. the Rectangle, under AB and BC, is equal to that under DB and BE. Conſ. as DB : AB, or BC :: BC, or AB : BE.- 9.6. El. Therefore, BE is a fourth Proportional.

If a third Proportional be required, AB and BC muſt be each equal to that Term, of two given Lines, which you require to be the middle Term of the three.

For, if three Quantities are Proportionals, the middle Term is a Mean between the other two.

In

In analogous or equal Proportion of four Quantities, ſince the firſt has the ſame proportion to the ſecond, as the third has to the fourth; and conſequently, the firſt is to the third as the ſecond to fourth; a Rectangle under the two extreme Terms (in equal Ratios of Right Lines) is equal to a Rectangle under the two middle Terms (9. 6. El.); which, in Numbers, is eaſily proved.

Take any four proportional Numbers, either continual; e. g. as 3, 6, 12, and 24; or in equal Ratios, as 3, 5; 9, and 15.

Now, it is plain that the firſt Number, 3, has the ſame proportion to the ſecond, 5, as 9, the third, has to the fourth, 15.

For, if 3 be multiplied three times, it is equal 9; and 5 multiplied three times, is equal 15; conſequently, 3 has the ſame proportion to 9, the third Number, as 5, the ſecond, has to 15, the fourth; and ſo it will ever be, when any two Numbers are multiplied equally.

As 4 is to 6, ſo is 10 to 15; conſequently, 4 : 10 :: 6 : 15.

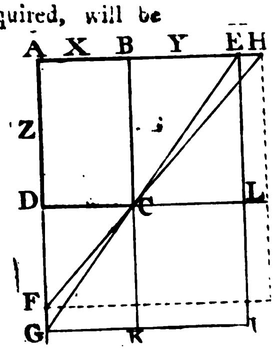
Now, in both theſe Caſes, the two extreme Terms, i. e. the firſt and the laſt, viz. 4 and 15, and the two middle Terms, 6 and 10, remain the ſame; only, the middle Terms have changed places; but, the ſecond multiplied by the third or the third by the ſecond is the ſame thing; and is always equal to the fourth multiplied by the firſt, or the firſt by the fourth.

Wherefore, if four Lines are proportional, as above; a Rectangle under the firſt and the fourth, the two Extremes, is equal to a Rectangle under the two mean or middle Terms; that is, the Rectangles have equal Areas, ſeeing, the Area, of a Rectangle, is produced by the multiplication of one Side by the other.

Hence, a fourth Proportional may very readily and accurately be found; as follows.

Let X, Y and Z be three given Lines.
It is required to find a fourth Proportional, which ſhall have the ſame Ratio or Proportion to Z, as X has to Y.

In this Caſe, X and Z will be the two Extremes; for ſince X is leſs than Y; conſequently, the Proportional required, will be leſs than Z, and is, properly, a Mean.



Make a Rectangle, ABCD, under the two given Lines X, and Z.

Produce any two Sides, from the ſame Angle, as AB and AD; on either of which, make BE or DF, equal Y.

Draw EG or FH, through the Angle C, cutting the other Side, produced, in G or H; then is DG, or BH, the Proportional ſought.

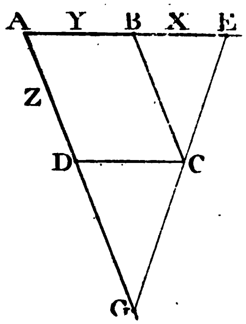
For, compleat the Rectangle AEIG.
Produce BC to K, and DC to L.

DEM. The Rectangle CI, is under BE (equal Y) and DG, the fourth Proportional required.

AC and CI are Complements of the Par. AEIG. Def. 38.

But, the Comp. in every Parallelogram are equal. - 19. 1.

Th. the Par. or Rect. CI is equal to the Rectangle ABCD.



But, if the Proportional was required to be to Z, as Y to X; then the Rectangle, or any Parallelogram, ABCD, must be under the two Lines Y and Z; which will, in this Case, be the two Means; and DG the Proportional sought, is now one of the Extremes; being the greatest of the four; which, in the former Case, was one of the middle Terms.

Thus may a fourth Proportional be found, either greater or less than either of the three given Lines, X and Z.

For, if a less Proportional was required, which should be an Extreme of the four; the Rectangle or Parallelogram, must be made under X and Y.

SCHOL. *When three Lines are given; a fourth Proportional is generally understood, to be either greater than the greatest, or less than the least of the three; but if another mean Proportional is required, to three Lines given, as it must be between the two Extremes of the three; so it will be, either greater or less than the middle Line, as that is either greater or less than a true Mean, between the other two. And if it be already a true Mean, there can no other be found, on either Side; for, the Square of a mean Proportional, being equal to the Rectangle under the two Extremes* (C. 2. 6.) *consequently, no other Rectangle but a Square, which shall have the true Mean for its Side, can be equal to a Rectangle under the two Extremes.*

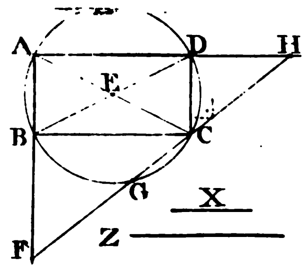
PRO-

PROBLEM XXXIII.

To find two mean Proportionals, to two given Lines.

Let X and Z be the two given Lines.

It is required, to find two other Lines; which not only contain an equal Rectangle, but, are in continual Proportion.



Construct the Rectangle ABCD, on the two given Lines; and on its Center, E, describe a Circle, circumscribing it. Produce the two Sides AB & AD, indef. Apply a Ruler to the Angle C, and move it on that Point, till it makes FC &. GH equal; and draw FH through G and C. Then will BF and DH be the two Proportionals sought.

For, DC, DH, BF, and BC are in geometrical Progression. And, a Rectangle under the two Means, DH and BF, is equal to the Rect. ABCD, under the two given Lines, X & Z.

Or, briefly thus.

Make a Right Angle FAH; in which, from the Angle A, make AB & AD respectively equal to the given Lines, X & Z. Join BD; which bisect in E; and, on E, with the Radius EB, describe a Semicircle, BCD.
Make DC equal AB (eq. X) and apply a streight Ruler to the Point C, making EF equal to EH, and it is done. Q.E.F.

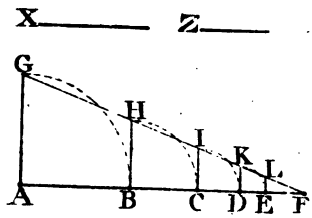
N. B. This method, though very ingenious, is not perfectly geometrical; seeing, the Points F and H cannot be ascertained, but by trial; yet, it is the best I have met with; for those methods which are performed by an Instrument, are not practical otherwise, and consequently cannot be called geometrical.

The method for finding a third Proportional (Pr. 31.) exhibits the reverse of this; for AB and BC, the two given Lines, in that Problem, are two Means, between the two Extremes BE and BD. For, BD : BC : : BC : BA : : BA : BE. - - - - P. 7. 6.

PROBLEM XXXIV.

To continue a progreſſive Proportion, between two given Lines, infinitely; and to repreſent the ſum of them all.

X and Z are the two given Lines.



Draw an indefinite Right Line, AF; in which, take AB equal to X, and BC equal Z.

Make BAG a Right Angle; and draw BH parallel to AG.

Make AG equal AB, and BH eq. BC.

Through the Points G and H, draw a Right Line, cuting AC, produced, at F; and AF is the whole Sum of the infinite Proportionals.

Draw CI par. to BH; make CD equal CI, and draw DK.
Make DE equal DK, and draw EL, perp. to AH.
Then is CI a third, DK a fourth, and EL a fifth Proportional; and, after the ſame manner, it may be continued, *ad infinitum.*

For, ſince AG is leſs than AF; ſo BH is leſs than BF; and conſequently, CI, DK and EL, will ſtill have the ſame Ratio to CF, DF and EF; wherefore, the laſt may always be taken from the remainder, and therefore, AF, is equal to the whole ſum of the infinite Proportionals; and AB, BC, CD, &c. or, AG, BH, CI, &c. are in a progreſſive, geometrical Proportion.

The Demonſtration of this Problem and the laſt are given at the end of the ſixth Book.

PRO.

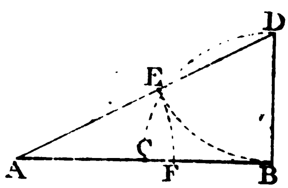
PROBLEM XXXV. 11. II. Euclid.

To divide a Line in extreme and mean Proportion.

AB is the given Line.

It is required to cut it so, that the lesser Segment shall have that Proportion to the greater, as the greater Segment has to the whole Line.

Or, the Rectangle under the whole Line and the least Segment shall be equal to the Square of the greater.



Bisect AB, in C; draw BD perpendicular to AB and equal to BC, half the given Line, and draw AD.
On D, with the Radius DB, describe an Ark BE, i. e. make DE equal DB; and, on the point A, describe the Ark EF, cuting AB in F, the Point sought, Q. E. F.

N. B. AF, the greater Segment, is the difference between the least Side, DB, and the Hypothenuse, AE, of the Right-angled Triangle ABD; constructed on the whole given Line, and half the Line, for the Base and Perpendicular. - - - - Pr. 13.

This Problem is otherwise performed and demonstrated in Prop. 11. of the second Book of Elements.

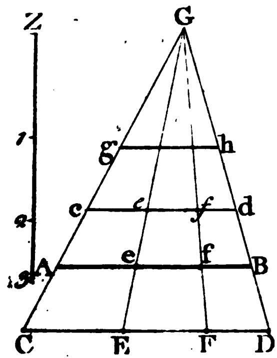
PRO-

PROBLEM XXXVI. 10. VI. Euclid.

To divide a given Line, in any known Proportion.

AB is the given Line; and Z, is a Line divided in the Proportion required, at 1, 2, and 3.

At any diſtance from AB, at diſcretion, draw CD parallel to AB; and make CE, EF, and FD equal, reſpectively, to the Diviſions on Z. Through the extreme Points of the two Lines, AB and CD, draw CA and DB, produced till they interſect, at G.



Draw EG and FG, cuting AB in e and f.

Then, is the given Line, AB, divided (in e and f) in the ſame Ratio as CD (in E and F) or as the given Line, Z, is divided, in the Points 1, 2, 3. - - - - P. 2. 6. El.

If the Meaſure given had been leſs than the Line given to be divided (as in this Example it is greater) AB would be divided the ſame.

Let cd, leſs than AB, be divided in the known Ratio, *e* & *f*. Draw, as before, the Lines Ac, and Bd, through their Extremes, meeting at G; and, from G, through the Diviſions *e* and *f*, draw the Lines G*e*, G*f*, cuting the given Line, AB, in the ſame Points e and f. Q. E. F.

The difference, it is evident, is only in the operation, for the effect is the ſame.

In the firſt Caſe, when the Meaſure is greater than the Line to be divided; then, the Vertex G will fall on the oppoſite Side, from CD; but, when it is leſs, as cd, the Vertex, G, will fall on the ſame Side with cd; as is obvious from the Figure.

N. B. Any

N. B. Any other Line drawn between the two Lines GC and GD, as gh, parallel to AB or CD, will be divided in the same Ratio, by the Lines GE and GF.

2. The same thing may be very readily and accurately done after this manner.

AB is the given Line, to be divided; and HI is a Line divided in the given or known Ratio, in 1 and 2.

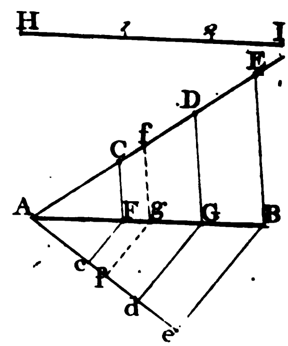
At either extreme of the given Line, draw AE, in any Angle at pleasure.

An acute Angle, not too small, is best.

Make the divisions at C, D, and E, equal, respectively, to the divisions on HI; according as you require them on AB, begin at either end, H or I.

Join the Point E, and the other extreme, B, of AB; draw DG and CF, parallel to EB, cuting AB in G and F.

So shall AB be divided in the same Ratio, as HI, in F and G, - - -. P. 2. 6.



If the Measure given had been less than AB, as Ae, divided in c and d, the Divisions, on AB, would be the same,

N. B. If it was required to divide AB into any number of equal Parts; make so many equal Divisions on CD, in the first, or on AE, in the last method, at pleasure; and proceed as directed.
AB may be readily bisected by either method; by the last, making Af, fE, or fe, two equal divisions, and drawing fg, as before, parallel to EB or eB.

The first method is demonstrable, from Prop. 6. 6. & Corol.
For the Triangles CGe and AGe, also EGF and eGf, &c, are similar. Conf. Ae : ef : : CE : EF, and as fB to FD.

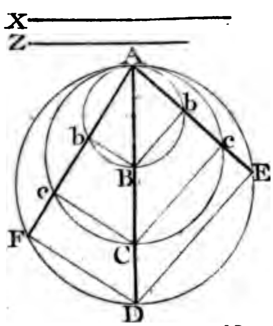
The last method is easily deduced from Prob. 32, and the Demonstration of it, from Prop. 2. 6. El.

APPL. It is almoſt needleſs to give an Application of this Problem, as it ſpeaks its uſe ſufficiently. It is extremely uſeful in the practice of Perſpective, as well as in all geometrical Drawings; as Plans, Elevations, &c.

AB is conſidered as a finite Line, already drawn, in ſome Plan, &c. and the Ratio or Proportion given, is certain known meaſures, or diviſions, to be repreſented on AB, from either a greater or leſs Scale of Proportion.

3. There is another method, by which a Line may be divided from a greater Meaſure given; which, not ſo much for its utility as the ſingular elegance of it, I ſhall give, as follows.

Let AD be divided in the given Ratio, at B and C; and let X or Z be a Line given, to be divided, in equal Ratio.



Deſcribe three Circles, on the three Diameters AB, AC, and AD.

Take the Line X or Z in your Compaſſes; and, ſeting one Point in A, cut the greateſt Circumference at E or F, with the other Point, and join AE or AF; which, will be divided in the ſame Ratio as AD, in b & c.

Draw Bb and Cc.

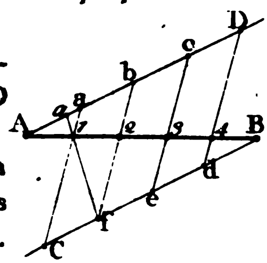
DEM. The Angles AbB, AcC, alſo AED or AFD, are Right. - - - - - - P. 12. 3. El. wherefore, AbB, AcC, &c. are ſimilar Triangles, the Angle at A being common to them all. - - C. 3. 2. 6. Therefore, AE or AF is divided in the ſame Ratio as AD, the Lines Bb, Cc, and ED, or FD, being parallel.—2. 6.

4. A Right Line may be accurately divided into any number of equal Parts, by the following Method.

Let AB be the given Line, to be divided into five equal Parts.

From

From either extreme, as A, draw AD, making any Angle, BAD, at pleasure; and, at the other extreme, B, make the Angle ABC equal BAD.



On AD and BC take, at pleasure, four equal Divisions, from A and B, at a, b, c, D and, d, e, f, C.

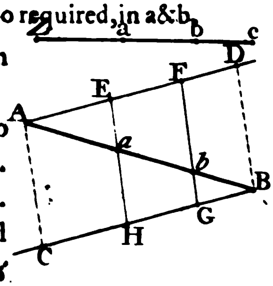
Join, a C, b f, &c. as in the Figure, which will divide AB into five equal Parts, as required; at 1, 2, 3, and 4. Q. E. F.

This is very evident from the second method; for, the Divisions on A D and B C being equal, they are consequently equal on AB; the Lines A C, b f, &c. being parallel (C. 15. 1.) for, AD is parallel to B C; by P. 4. 1.

N. B. If the Divisions on A D and B C were either greater or less, AB would be divided the same; which is obvious by taking A *a* 3 fourths of A a, and joining a f; for, B f is also 3 fourths of B C.

I have never seen this method used for dividing a Right Line in any given Ratio; which may be applied with success.

Let AB be a Line given to be divided; and Z, a Measure known, i.e. a Right Line divided in the Ratio required, in a & b.



Make AD and BC in equal Angles with A B, as before.

Transfer the measures Z a, a b, and b c, to AD and B C, in the order required; viz. make AE equal Z a, and E F equal a b. also, make B G equal b c, and G H equal a b, &c. and join the Points E H, F G; which will divide AB, in the Ratio of Z, in the Points *a* & *b*.

Note. The two extreme Divisions FD and C H are of no use in the operation; but AC and DB, being joined, will be parallel to EH & FG; and exhibits the former Method, applied on both Sides.

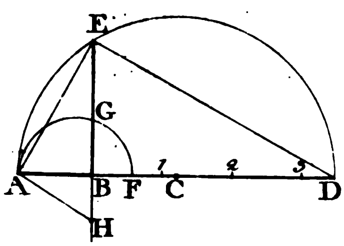
The only difference is, that, in this Method, there is no regard had to the parallelism of the Lines E H and F G, but only to join the Points, and they are necessarily parallel.

PROBLEM XXXVII.

To find the Side of a Square, or any other Right-lined Figure (ſimilar to a given one) which ſhall be to the given Figure in any proportion required.

Let AB be the given Line, or Side of the given Figure.

It is required to find the length of a Line, for a correſponding Side; on which, if a ſimilar Figure be conſtructed, its Area ſhall be to the given one in the Ratio of $3\frac{1}{2}$ to 1.



Produce the given Line, AB, indef. Make BD equal to $3\frac{1}{2}$ times AB. Biſect AD; and, on the point of biſection, C, deſcribe a Semicircle, or the Ark AE only. At the Point B, draw BE perpendicular to AB, cuting the Ark at E; BE is the Line ſought. Q. E. F.

DEM. For, BE is a mean Proportional between AB & BD. Wherefore, the ſquare of AB, to the ſquare of BE, is duplicate of AB to BE. - - - - P. 10. & 12. 6. i. e. their Ratio or Proportion, to each other, is as AB to BD; and all ſimilar Figures are in the ſame Ratio, as the Squares of their correſponding Sides. - - C. 2. 13. 6.

By this Problem, any right-lined Figure, whatever, may be increaſed or diminiſhed in any Proportion. e. g.

If you would decreaſe it a fifth, a fourth, a third, or a half, &c. make BF to AB in that Ratio; biſect AF, and deſcribe the Semicircle AGF, cuting the Perpendicular BE (at the Point B) in G; and BG is the Line ſought.

Hence the Ratio between any two Figures, may be known.

For,

For, having, by Prob. 24. and 25. reduced the given Figures to Squares; i. e. having found the side of a Square, equal to each Figure, respectively, make AB and BE respectively equal to them, and forming a Right Angle, ABE.

Produce AB or EB, indefinite; join AE, and make AED, or EAH, a Right Angle; i. e. draw ED, or AH, perpendicular to AE, cuting AB, or EB, produced, in D or H.

Then, as AB is to BD, or, as HB to BE, so is one Figure to the other;—by Cor. 1. P. 13. 6. El.

For, the Squares of AB and BE are, respectively, equal to those Figures.

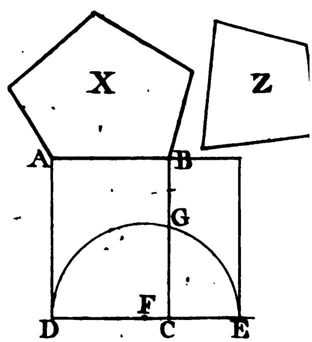
N. B. If the Figures, whose Ratio is required, are similar, make AB and BE, respectively, equal to any two corresponding Sides, and proceed as above.

PROBLEM XXXVIII. 25. VI. Euclid.

To construct a Poligon similar to a given one, and equal to any given Right-lined Figure.

X and Z are the two Figures given.

It is required to make a Pentagon similar to X, and equal to the Trapezium Z.



On any Side of the Pentagon, as AB, make a Rectangle ABCD equal to the Pentagon; by Prob. 24, 20, and 23. Then, on the Side BC, of the Rectangle BD, make another Rectangle, BE, equal to the Trapezium; by the same Problems. Bisect DE, in F; and, on that Center, describe a Semicircle, with the Radius FD, cuting BC, at G.

On CG, if a Pentagon be constructed, similar to X, making CG the corresponding Side to AB, it will be equal to the given Trapezium Z. Q. E. F.

DEM. For

DEM. For, CG is a mean Proportional between DC & CE. Wherefore, since the Ratio of any two similar Figures, is duplicate of their corresponding Sides (13. 6.) and the Ratio of DB to BE is as their Bases, DC to CE; - 1. 6. Conf. the Ratio of the two Pentagons, will be as DC to CE; for, they are respectively equal to the Rectangles DB & BE. Therefore, a Pentagon constructed on CG, similar to X, will be equal to the Trapezium Z. Q. E. D.

By the 24th any Figure is readily reduced to a Triangle, by the 20th a Rectangle is found equal to a Triangle; and by the 23d, another Rectangle may be found equal to that, having one Side equal to AB, or any given Line.

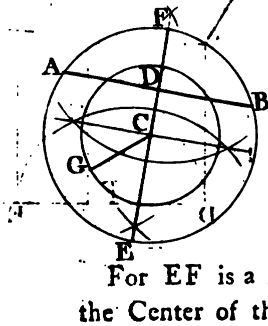
For, it will produce AD, for the other Side of the Rectangle AC, which is a fourth Proportional to AB and the two Sides of that Rectangle, found by the 20th, equal to the Pentagon; as by the last method of Prob. 32.

Conf. the Rectangle BD is equal to the Pentagon X, and BE to Z.

PROBLEM XXXIX. 1. III. Euclid.

To find the Center of a given Circle; and through a given Point, to describe a Circle parallel to the given one.

First. ABE, is the Circle given; and G the given Point.



Draw a Chord Line, AB, at pleasure. Bisect AB (Pr. 8.) and through D, the point of bisection, draw EF perpendicular to AB (6.) cuting the Circumference in E & F. Bisect EF, in C, which is the Center of the Circle ABE. Q. E. F.

For EF is a Diameter, and consequently passes through the Center of the Circle. - - - - - - P. 1. 3.

2nd. Join

2nd. Join CG. With the Radius CG, and on the Center C, describe a Circle GH, and you have done. Q. E. F.

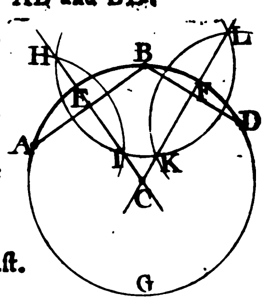
SCHOL. *All parallel Circles, in the same Plane, have the same Center, and are called concentric; but, if they are in parallel Planes, as on a Cylinder, Cone or Sphere; a Right Line perpendicular to those Planes, if it passes through the Center of one Circle, it will pass through the Centers of them all; which line is the Axis of the Cylinder, Cone or Sphere.*

PROBLEM XL. 25. III. Euclid.

To perfect or compleat a Circle, from a given Ark or Segment of that Circle.

ABD is the Ark given.

Draw, at pleasure, two Chord Lines; AB and BD. Bisect the two Chords, at E and F; from which, draw the Perpendiculars EC and FC, intersecting at C. - - Pr. 8. & 6. Then will C be the Center of the Circle; on which, with the Radius CA, CB, or CD, the Circle ABDG may be compleated



This Problem is demonstrated in the last.

N. B. By this method, may be found the Center of a perfect Circle, as readily as in the foregoing; nor is it necessary to draw the Chords; only, assuming, at pleasure, three points, A, B, & D, and proceed after the manner following.

With any Radius, at discretion, on B, as a Center, describe the Ark HIKL; and, with the same Radius, on A and D, describe two Arks, HI and KL, cuting the former in H, I, K and L; through which Points, draw Right Lines, HI and LK, intersecting at C, the Center sought.

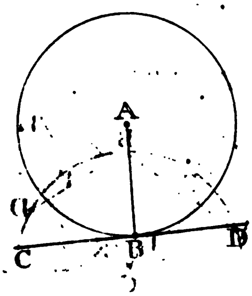
COR.

COR. Hence, through three given Points (not lying in a Right Line) as A, B, and D, the Circumference of a Circle may be defcribed. whofe Center (being found, as above) is C.

PROBLEM XLI.

To draw a Tangent to a Circle, through a given Point in the Circumference. And, to find the Point of Contact of a Tangent to a Circle.

Firft; B is the Point given; through which a Tangent is required to be drawn.



Having found A, the center of the Circle (by the foregoing) join the point B, and the center of the Circle by a Right Line, AB. At the Point B, make a Right Angle, ABC, and produce CB, towards D. The Right Line CD will touch the Circle, in the Point B. - - P. 8. 3. El.

2nd. CD is the Tangent given; to find the Point of Contact.

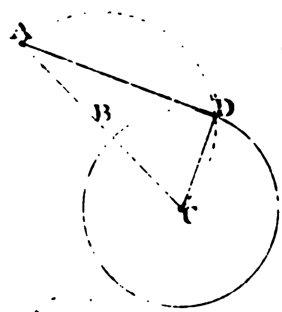
Draw a Perpendicular AB to the Tangent, from the center of the Circle; cuting CD in B; - - - - Pr. 7.

B is the Point of Contact, in which the Tangent, AD, touches the Circle. - - - - - - C. 3. 8. 3.

PRO.

PROBLEM XLII. 17. III. Euclid.

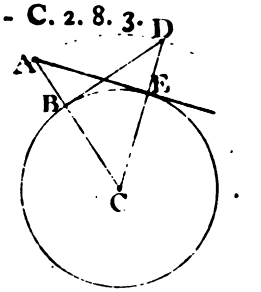
To draw a Tangent to a given Circle, from a Point given without the Circle; i. e. to determine the Point, in which a Right Line drawn from the given Point ſhall touch the Circle.



From the given Point, A, draw AC; to the Center of the Circle.
Biſect AC; and, on the point of biſection, B, with the Radius AB, deſcribe a Semicircle; cuting the circumference of the given Circle in D, the Point ſought. Q.E.F.

DEM. Having drawn AD and CD, the Right Line AD will touch the Circle at D.
For, the Angle ADC is a Right one. - - P. 12. 3.
Wherefore, AD touches the Circle in D. - - C. 2. 8. 3.

Otherwiſe.



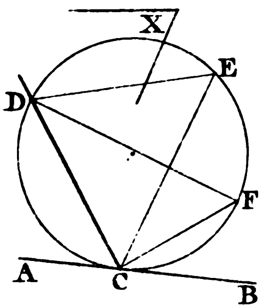
Join the Point A, and the Center C, as before, cuting the Circumference in B.
With the Radius CA deſcribe the Ark AD.
Draw the Perpend. BD, cuting AD at D.
Laſtly, draw CD, cuting the given Circle at E, the Point ſought. Draw AE.
The Right Line AE will touch the Circle, at E.

DEM. The Triangles AEC, BDC are congruous.
For, the ſides AC, CE are equal to DC and CB, reſpectively; and the Angle C is common to both; therefore, AE is equal to BD; the Angle at A equal D, and the Angle AEC equal DBC. - - - - - - P. 8. 1. El.
But CBD is a Right Angle (Con.) wh. AEC is a R. Angle.
Therefore, AE touches the Circle at E. - C. 2. 8. 3.

PROBLEM XLIII. 34. III. Euclid.

To cut off a Segment of a Circle, which ſhall contain an Angle equal to a given one.

DEC is the Circle given, and X the given Angle.

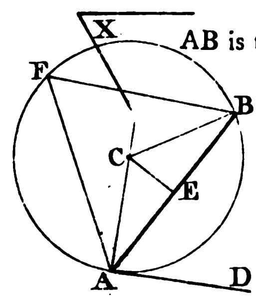


Draw a Tangent, AB, to the given Circle. At the Point of contact, C, make the Angle ACD equal to X, the Angle given; and DEFC is the Segment required. Q.E.F.

DEM. For, if from any Point in the Circumference, as E or F, Lines are drawn to the extremes of the Chord CD; the Angle DEC or DFC, is equal to ACD, (equal X, by Con.) - - - P. 13. 3.

PROBLEM XLIV. 33. III. Euclid.

On a given Line, to deſcribe a Segment of a Circle, which ſhall contain an Angle equal to a given one.



AB is the given Line, and X the given Angle.

Make an Angle, BAD, equal to the given Angle, X. - - - - - Pr. 4. Draw AC perpendicular to AD, - - 10 and, on the other Extreme, B, make the angle ABC equal BAC.

Or, having biſected AB, draw EC perpendicular to AB, cuting AC in C; on which, with the Radius CA or CB, deſcribe the Ark AFB, the Segment required.

Or

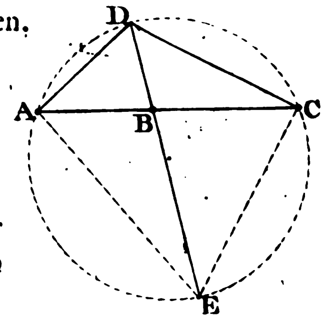
DEM. From any Point, as F, draw FA and FB.
The Ang. AFB=BAD, eq. to the given Ang. X.-13.3.

Or, without drawing AD, on the given Line AB, make an Iſoſceles Triangle, whoſe Angles at the Baſe, BAC, ABC, are each equal to the Complement of the given Angle, to a Right Angle.

The Vertex C will be the Center, and CA, or CB, the Radius of the Circle required.

PROBLEM XLV.

The Angles being given, under which three Objects, ſituate in a Right Line, are ſeen, and their Diſtances from each other known; to determine the Point from which they are ſeen.



A, B, and C are the three Objects.

Make the Angles ACD, CAD, alternately, equal to the given Angles, i. e. make ACD equal to the Angle under which AB is ſeen, and CAD equal to the other Angle given.
Produce CD and AD, interſecting at D.
Deſcribe a Circle through the two extreme Objects, A & C, and the Angle D; - - by Pr. 40.
Draw DB, and produce it, till it cuts the oppoſite Circumference, at E, the Point ſought.

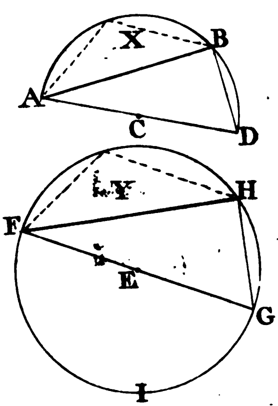
DEM. For (having drawn AE and CE) the Angle AEB=ACD, and the Angle CEB=CAD. - P. 10. 3.

PROBLEM XLVI.

A Circle being given, to cut off a Segment, ſimilar to a given Segment; and, on a given Line, to conſtruct a Segment, ſimilar to a given one.

Firſt. It is required to cut off, from the given Circle FHI, a Segment ſimilar to X.

Find the Center, C, of the given Segment, X, - Pr. 40. and compleat the Semicircle ABD.



Through the Center, E, draw FG, and make the Angle GFH equal DAB; - Pr. 4. The Segment, Y, cut off by the Chord FH, will be ſimilar to the given Segment.

DEM. For, the Angle GFH$=$BAD.-Con. And (having joined BD and HG) the Angle FHG$=$ABD - - - Ax. 9. (for they are Right Angles; 12. 3.) conſ. FGH$=$ADB - - C. 5. 10. 1. wherefore, the Triangles ABD, FHG are ſimilar. - - - C. 2. 4. 6. Therefore, as FG : AD :: FH : AB.-4. 6. And, by taking away the Triangles ABD, FHG, from the Semicircles, there is left the Segment Y ſimilar to X.

N. B. If the Segment given had been greater than a Semi circle, the operation is the ſame; the Triangle GFH being added to the Semicircle FIG, inſtead of taking it away from FHG.

2nd. FH is the given Line, on which to conſtruct a Segment as required.

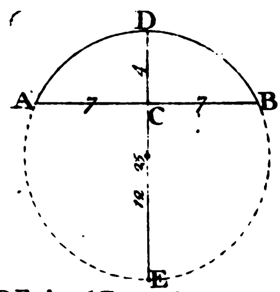
Make the Angle HFG equal to BAD; which, AB, the Chord of the given Segment, makes with the Diameter AD; found as above.
Draw HG perpendicular to FH, cuting FG in G.
Biſect FG, and on E, the Center, with the radius EF or EG, deſcribe the Ark FCH, forming the Segment required.

APPL. This Problem is very uſeful to Builders, to form Arches, and ſcheme Heads for Windows, Doors, &c. ſimilar to others, either greater or leſs, of any proportion required.

The following is an univerſal Rule, to find, arithmetically, the Diameter of a Circle, from any given Segment, by the meaſure of the Chord Line or Subtenſe; i. e. the Baſe of the ſegment, and the height of the Ark.

ADB is the Segment given.

Divide the Square of half the Baſe by the perpendicular height and add the Quotient to the Diviſor. See Caſe 2.14.3.



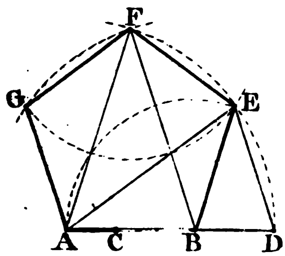
e. g. Let the Baſe AB be 14 and the Perpendicular CD, 4.
The Square of AC or CB, 7 multiplied by 7 is equal 49; which, divided by 4, gives the Quotient 12, 25, equal CE; to which add the diviſor, 4, the height of the Ark, CD, it gives 16, 25, for the Diameter.
For a Rectangle under CE and CD is equal to the Square of AC. - - - - P. 14. 3.

Suppoſing the Integers Feet, the Diameter, DE, is 16 Feet 3 In.

The Rule is the ſame, if the Segment be greater than a Semicircle. But, having found the Center (by Pr. 39 or 40) a Right Line drawn through the Center, cuting the Ark in two Points, is a Diameter.

PROBLEM XLVII.

To make a regular Pentagon, on a given Line, AB.



Divide AB in extreme and mean Proportion, in the Point C. - - Pr. 35. Produce AB; and make BD equal to BC, the greater Segment; then, AD is to AB, as AB to BC, and as BC to AC.*

On A, with the Radius AD, defcribe the Ark DEF; and, on B, defcribe the Ark GF, interfecting at F.

On F, with the Radius AB, defcribe the Ark EG; i. e. make FE and FG each equal AB, and join the Points A and G, B and E; alfo E, F and F, G, which compleats the Pentagon. Q. E. F.

The Demonftration of this conftruction of a Pentagon may be obtained, from the 7th and 8th Prop. of the 4th of Elements.

For, AED is an Ifofceles Triangle, having its Angles at the Bafe, ED, each double the Angle EAD, at the Vertex; AFB is the fame, which may be confidered as infcribed.

And, by 34. of the 6th. the Diagonal of a Pentagon, BF or AE (equal AD) has that ratio to the Side, AB, BE, &c. as the greater Segment to the lefs, of a Line divided in extreme and mean Proportion.

Or it may be conftructed thus.

Having found the Point D, as above, and drawn the Ark DEF, on the Center A.

On B, with the Radius AB, defcribe the Ark AE, cuting the other at E, and draw AE and DE.

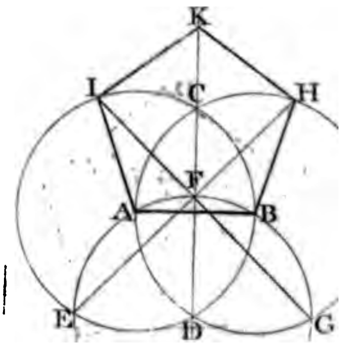
Draw BF parallel, to DE; and alfo AG, indefinite.

Draw FG parallel to AE, cuting AG in G, and join FE and BE, which compleats the Figure.

* See N. B. 2. Prop. 35. 6th of Elements.

Otherwife,

Otherwise, mechanically; AB being the given Side.



On A and B, with the Radius AB, describe two Circles, cuting each other in C and D, and draw CD.

On D, with the same Radius, describe the Ark EABG, cuting the two Circles and the Right Line CD, in E, F, and G. Draw the Right Lines EF and GF, and produce them till they cut the Circumferences, in H and I.

Then, on H and I, with the Radius AB, describe two Arks, interfecting at K, and join the Points AI, IK, KH, and HB, which compleats the Pentagon, AIKHB. Q. E. F.

Of this construction of a Pentagon, there is no Demonstration has yet been given.

I shall, here, give a general Method for constructing every kind of regular Poligon, from a Pentagon to a Duodecagon, by means of the following Table; in which, the Angle of the Poligon is determined by the proportion it has to a Right Angle, and the difference there is between them.

A Right Angle is to the Angle of a Poligon as follows,

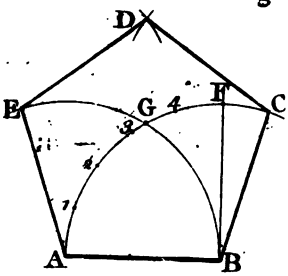
	Ratio	Diff.		Ratio	Diff.
Pentagon	as 5 to 6	— 1	Nonagon	as 9 to 14	- 5
Hexagon	as 3 to 4	— 1	Decagon	as 5 to 8	- 3
Heptagon	as 7 to 10	— 3	Undecagon	as 11 to 18	- 7
Octagon	as 2 to 3	— 1	Duodecagon	as 3 to 5	- 2

Note; the Hexagon, Octagon, &c. having an equal number of Sides, are reduced to the lowest Denomination; otherwise, the Right Angle is always supposed to be divided into the same number of Parts as the Poligon has Sides; which will then be, for a Hexagon, as 6 to 8, difference 2; in an Octagon, as 8 to 12, difference 4; and so of the others.

To

To conftruct a Pentagon, by the Table, on the Line AB.

Make a Right Angle, ABF, on the extreme Point B.



By the Table, the Angle of a Pentagon is to a Right Angle in the ratio of 6 to 5, difference 1.

With the Radius AB (or any other) on B defcribe an Ark from A to C.

Divide the Ark (of the Right Angle) AGF into five equal Parts, as in the Figure; the Angle of the Pentagon is fix of thofe Parts.

Add the difference, 1, from F to C, and draw BC.

The Angle ABC is the Angle of a Pentagon, containing 6 fifths of a Right Angle, on the Ark AGC.

On A, with the fame Radius, AB, defcribe the Ark BGE, cuting the other at G; make GE equal GC, and draw AE.

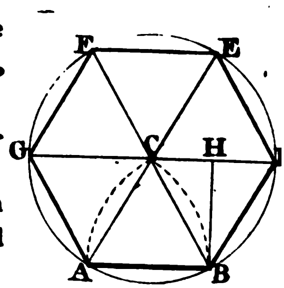
With the fame Radius, on E and C, defcribe two Arks, interfecting at D, and join CD and DE; which compleats the Pentagon.

DEM. The Sides AB, BC, &c. are equal, by Conftruction. And the Angles are alfo equal, being fubtended by equal Arks, AGC, BGE, &c. of equal Circles. - C. 2. 9 3. Therefore, ABCDE is a regular Pentagon.

PROBLEM XLVIII.

To make a Hexagon, on a given Line, AB.

In the conſtruction of a Hexagon you need not regard the Table, nor proceed after that method; ſeeing that, the Side of a Hexagon is always equal to the Radius of a circumſcribing Circle. - - - - - - 11. 4. El.



Therefore, with the Radius AB, deſcribe two Arks, on A and B, interſecting at C, and compleat the equilateral Triangle ACB. On the Center C, with the ſame Radius, deſcribe a Circle.
Produce AC to E, and BC to F; and, through the Center C, draw GD parallel to AB, and join the Points A, G, F, E, D, and B.

Or, having found the Center C, and deſcribed a Circle, as above.

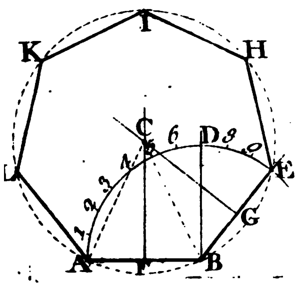
Apply the given Side, AB (equal to the Radius, AC) ſix times round the Circumference, from B to D, E, F, G, and join the Points as before.
AGFEDB is a regular Hexagon. Q. E. F.

The circumference of a Circle contains the Radius exactly ſix times inſcribed (Th. P. Ang. Art. 5.) conſequently, it is equal to the Side of a Hexagon inſcribed. - P. 11. 4.

This needs no other Demonſtration; for, the Sides are all equal by Conſtruction; and the Triangles ACB, BCD, DCE, &c. are Equilateral, whoſe Angles are alſo equal (C. 1. 9. 1.) Each Angle of the Hexagon, contains two Angles of a Triangle, ABD, equal ABC added to CBD, &c. which are therefore equal; and conſequently have that proportion to a Right Angle as 4 to 3; as the Perpendicular BH, biſecting the Angle CBD, indicates.

PROBLEM XLIX.

To make a Heptagon, on a given Line, AB.



Make a Right Angle ABD; and divide the Ark of it into seven equal parts, as many as the Figure has Sides.

Observe, by the Table, that the angle of a Heptagon is ten of those Parts, the difference is three.

Set off three Parts from the Perpendicular BD to E, and draw BE. Make BE equal AB. Bisect the two Sides AB and BE; - Pr. 8. and draw the Perpendiculars, FC and GC, intersecting at C. On C, the Center, and Radius CA, or CB, describe a Circle; which will contain the given Line, AB, seven times in its Circumfnrence, at A, B, E, H, I, K, L; which Points, joined by Right Lines, will compleat the Heptagon required. Q. E. F.

DEM. It is equilateral by Construction; and it is also equiangular, because inscribed in a Circle; for, equal Segments contain equal Angles. - - - P. 10. 3.

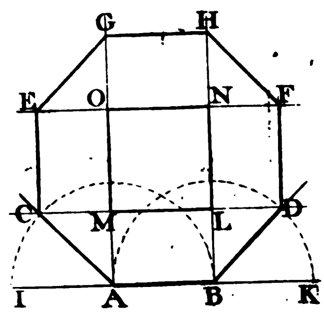
N. B. It is plain that the Angle ABE, which is an Angle of the Heptagon, is equal to the Right Angle ABD added to the Angle DBE; the Right Angle containing seven parts, the Acute Angle, DBE, three; and consequently, the Angle ABE contains ten; therefore, the Ratio is as 10 to 7; as by the Table.

After the same manner a Circle may be found, which shall contain a given Line any number of times, to twelve, applied to the Circumference, by the Table, given above.

PRO-

PROBLEM L.

To describe an Octagon, on a given Line, mechanically, without the Table.

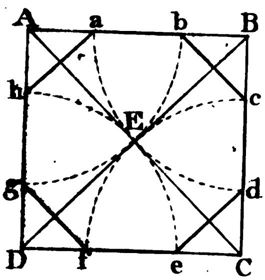


On the extremes of the given Line, AB, draw the Perpendiculars AG and BH, indefinite.
Produce AB, both ways, to I and K.
Bisect the external Angles, IAG & HBK, by the lines AC and BD; which, make equal to AB, and draw CD; which, will be parallel to AB.
Make LN equal LM; and, through N, draw EF parallel to CD.
Draw CE & DF, parallel to AG & BH.
Make NH equal NF, and draw GH parallel to EF; and join EG and FH; which compleats the Octagon. Q. E. F.

N. B. It is obvious, in this Figure, that its Angles, as ABD, has the proportion to a Right Angle, ABL, of 3 to 2, as in the Table; for, the Angle LBD is half the Right Angle LBK; and LBD added to ABL, equal ABD, is the Angle of the Octagon.

PROBLEM LI.

To find the Side of an Octagon, in a given Square, ABCD.



Draw the two Diagonals AC and BD, which gives the Center E. - P. 16. 1. El. With the Radius of half the Diagonal, AE, on every Angle of the Square, describe an Ark or Quadrant, bEg, aEd, &c. i. e. make Ab, Ag, Ba, Bd, &c. equal half the Diagonal; and, joining the Points ah, bc, &c. you will have a regular Octagon abcdefgh. Q. E. F.

APPL. The uses of these Problems, to various Mechanics, are very obvious. By the 50th we learn how to form an Octagon Building; as a Temple, Library, &c. an Alcove, or Bow Window, &c. (which are frequently half a regular Octagon or Hexagon) of a given measure, for a Side of the Building, &c.

By this Problem, we find the measure of a Side, when the width or Diameter is first determined.

To enumerate all the uses of Poligons, would be impertinent and foreign to the purpose; my design being to shew how to construct them, in the easiest and readiest manner, the application of them will readily occur, as occasions require.

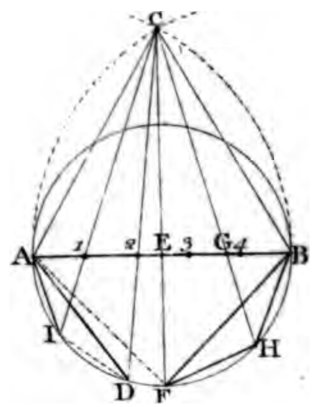
These are the most necessary Poligons for mechanical uses; and since, by the Table, and Rules already given, any Poligon, to twelve Sides, may be readily constructed, I shall defer treating more fully on them to the fourth Book; which teaches how to inscribe and circumscribe all kinds of regular Figures.

Although it is entirely Problematical, yet I think it best to follow the order of Euclid, in that; seeing, it cannot be properly treated on without the Elements of the first three Books. The practical Part is, nevertheless, easy and intelligible to any Capacity.

PRO-

PROBLEM LII.

To find the Side of any regular Poligon in a given Circle.



Draw a Diameter, AB, on which, construct an equilateral Triangle, ACB; or draw the two Arks only, interfecting at C. (This preparation is the fame for any Poligon whatever.)
Then, divide the Diameter into as many equal Parts as the Poligon, required, has Sides; (Pr. 36.) and, through the second divifion, from either extreme, draw a Right Line, from C to the oppofite Side, of the concave Circumference. e. g.

If a Pentagon be required, the Diameter (AB) muft be divided into five equal Parts; through the fecond Divifion, from A or B, draw CD; then, AD is the fide of a Pentagon; i. e. a fifth part of the whole Circumference, or two fifths of the Semicircumference AFB.

The Side of a Hexagon is equal to the Radius; and the Side of an equilateral Triangle is the Diagonal of two fides of a Hexagon; yet the fame Rule holds equally true in all.

If a Right Line, be drawn, from C, through the Center, E, to F, it divides the Circle equally into four, and AF or FB, is the Side of a Square.

For, if the Diameter be divided into four equal Parts, the Center being equally diftant from each extreme of the Diameter, AE is, confequently, two of thofe Parts.

One

One of thofe Parts, GB, equal half the Radius, EB, is two eighths of the Diameter; wherefore, if CG be drawn and produced, it will cut the concave Circumference in the Point H; and BH, or FH, being joined, is the Side of an Octagon; for it will bifect the Ark, FHB, of the Side of a Square.

Again. A1, being a fifth part of the Diameter, is equal to two tenths; wherefore, if C1 be drawn, to I, it will bifect the Ark AID; and AI or ID, being joined, is the fide of a Decagon, infcribed; for, it is equal to two tenths of the concave Circumference AFB.

Thus may the Side of any Poligon whatever, contained in a Circle, be obtained; by obferving the Rules given above. And it is truly worthy of notice; that, any Right Line, drawn from C, cuting the Diameter and the concave Circumference, will cut them both in the fame Proportion; or, in whatever Ratio one of them is divided, a Right Line being drawn, from C, through the point of divifion, will alfo cut the other in the fame Ratio.

Of this Conftruction, or equal divifion of the Diameter and the Circumference, no Demonftration can be given, having confulted feveral able Geometricians concerning it; who fay, that it is only an approximation and not mathematically true. Yet, I muft own, that I do believe it to be perfectly true, or it could never anfwer fo very accurately, as it does, in all Divifions, whatever.

OF

OF THE

ELLIPSIS.

HAVING now, gone through all the moſt uſeful and valuable Problems in Geometry; I ſhall, next, before I proceed to the Elements, ſhew how to deſcribe that uſeful and elegant Figure called an Ellipſis; which is ſo very neceſſary to Mechanics, and particularly to Architects and Builders, in general; inſomuch, that I ſhould reckon a compleat Syſtem of practical Geometry deficient without it.

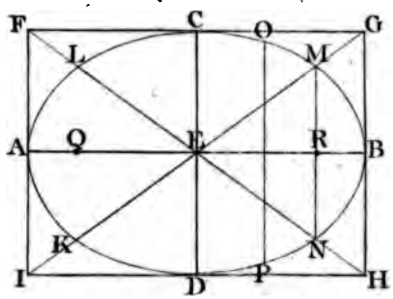
The Ellipſis, is a Figure which is not admitted into Plane Geometry; as the nature of its Curve and the properties peculiar to it, are of no uſe in the Elements of Euclid. Nevertheleſs, as it is a very uſeful Figure, I ſhall define all its parts, particularly, and ſhew how, by various ways, it may be deſcribed of any Proportion required; a thing much wanted and but little known, to many who have a conſtant occaſion for it; how many lame and imperfect Ovals may be ſeen amongſt the Works of our moſt eminent Artiſts, at an Exhibition, is but too obvious to a judicious Eye.

I ſhall alſo explain ſome of its peculiar Properties; ſhewing the affinity between the Circle and Ellipſis (as between a Square and Rhombus) the Ellipſis being conſidered as a Circle, preſſed gently, or drawn out, at the two extremes of any Diameter.

DEFI-

DEFINITIONS.

DEF. I. An ELLIPSIS, or OVAL, is a Plane Figure bounded by a regular curved Line, falling into itself, which, is not circular in any part, being described on two Centers.



DEF. II. PERIPHERY, or CIRCUMFERENCE, is the curved Line which bounds the Ellipsis; ACBD.

DEF. III. CENTER, of an ELLIPSIS, is the Point E, where any two Diameters intersect, and, consequently, bisect each other.

DEF. IV. DIAMETER, of an ELLIPSIS, is any right Line, as AB, MN, &c. passing through its Center, E, and terminated by the Periphery.

For, every such Line divides the Ellipsis equally in two; and is also bisected in the Center, E. Therefore, any two Diameters bisect each other; as in a Circle.

DEF. V. TRANSVERSE DIAMETER is the longest which can be drawn in an Ellipsis. As AB.

DEF. VI. CONJUGATE DIAMETER. This Term is generally confined to the shortest Diameter, CD; but, that is conjugate only in respect of the Transverse AB; which is also conjugate to CD.

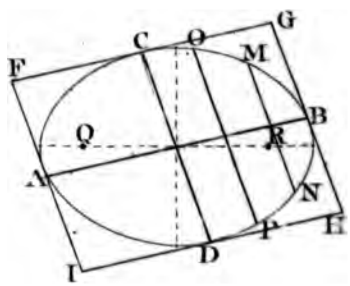
If Tangents to the Ellipsis be drawn, through the extremes, A and B, of the Transverse, and CD its conjugate Diameter, cuting each other in F, G, H and I, they

will

will form a Rectangle; and if the Diagonals of the Rectangle, FH and IG, be drawn, they will paſs through the Center of the Ellipſis; the parts, KM and LN, which are terminated by the Curve, are Diameters of the Ellipſis; each of which, is Conjugate, in reſpect of the other.

Theſe are the only two Conjugate Diameters which are equal.
The Tranſverſe Diameter, and its Conjugate, are the only two which are perpendicular to each other, or cut at right Angles.
The Tranſverſe Diameter, AB, biſects the acute Angles FEI, GEH; and the obtuſe Angles, FEG, IEH, are biſected by the Conjugate, CD.
Every Diameter, drawn within the acnte Angles, is greater than its Conjugate, which falls within the obtuſe Angles.

DEF. VII. ORDINATES are Right Lines drawn parallel to the Conjugate of any Diameter, as MN, OP, &c. and they are biſected by the Diameter, AB. The whole Lines MN, OP, are, therefore, double Ordinates to the Diameter AB.



DEF. VIII. A TANGENT is a Right Line touching the Periphery, at the extreme of a Diameter, parallel to its Ordinates. As FI, FG, &c.

DEF. IX. AXES, of an Ellipſis, are the Tranſverſe and its Conjugate Diameter; AB and CD, Fig. 1ſt.

DEF. X. FOCI, of an Ellipſis, are the Points Q and R, on which it is deſcribed; as a Circle on its Center.

DEF. XI. LATUS RECTUM, or RIGHT PARAMETER, is that Ordinate to the Tranſverſe Axe which paſſes through either Focus; as MN; Fig. 1ſt.

The Latus Rectum, and every Parameter, is a third Proportional to the two conjugate Diameters, to which it is the Parameter.

Note. The Parameter to the Diameters KM and LN, Fig. 1ſt. coincides with them; for, being equal, there cannot be a third Proportional to them.

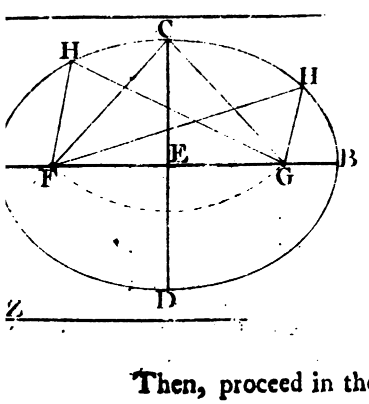
DEF. XII. ABSCISSA. If the Tranſverſe Diameter be cut, in any Point at pleaſure, the two Segments, made by that Section, are called Abſciſſas.

Note. An Ellipſis may be generated, by the Shadow of a Circle, or Ring, projected on a Plane, not parallel to the Ring; provided, the luminous Point is not in the Plane of the Circle.
Or it is an oblique Section, made by a Plane, of a Cylinder or Cone. (See Def. 15 and 18, 7th. El.)

PROBLEM I.

The Tranſverſe and its Conjugate Diameter being given, how to determine the Foci, on which an Ellipſis may be deſcribed, of the Proportion required.

Let X and Z be the meaſures given.



Draw, at right Angles, two Right Lines, AB and CD, equal, reſpectively, to X and Z, and biſecting each other, in the Point E.
On either extreme, C or D, of the Conjugate Diameter, deſcribe an Ark, with the radius AE, or EB, half the Tranſverſe, cuting it in F and G; which are the Foci, or Centers, on which the Ellipſis may be deſcribed.

Then, proceed in the manner following.

Take

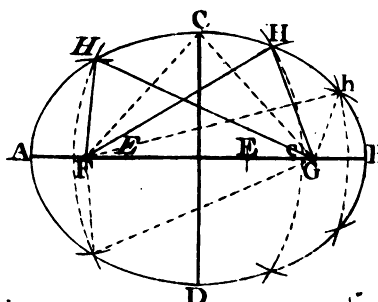
Take a fine, ſmooth Cord or ſtring, and, having fixed two Pins in F and G, carry the ſtring round both Pins, and draw it, on both Sides, to the Point C, or D; where, fix a Pencil, and revolve it around, on the two Pins, keeping the String at full ſtretch; the point C will deſcribe a Curve, which will paſs through the four Points A, C, B and D.

From which conſtruction it is evident, that any two Right Lines, FH and GH, drawn from the Foci to any Point in the Circumference, are equal to FC and CG; that is, to the Tranſverſe Diameter, AB.

For, FG being common to both the Triangles FCG and FHG, the remaining Sides, added together, are equal;
i. e. FC added to CG is equal to FH added to HG.

As it is difficult to keep the Point or Pencil true, in a ſtring, this Method is not ſo eligible, for ſmall Ovals, or for any, which require the Curve to be exact; but, from what has been advanced, (being well conſidered and underſtood) it will be found practicable to deſcribe ſmall Ovals with tolerable exactneſs.

Having determined the two Foci, F and G, in the given Tranſverſe Diameter, AB, whoſe Conjugate is CD (as above) as many Points, H, may be determined, in the Periphery, as are neceſſary; through which a Curve may be deſcribed, by a ſteady Hand, which will be a true Ellipſis; after the following manner.



With any Radius, at pleaſure, ſetting one Point of the Compaſſes in either Focus, as at F; with the other Point, make an Ark, at H. Then, with the ſame Radius, FH, ſeting one Point in either extreme of the tranſverſe Diameter, as at A, cut the Tranſverſe, at E with the other. Take the remaining Segment, EB, as Radius, and, on the Center, G, deſcribe another Ark, cuting the former in H; which will be a Point in the Periphery.

Thus, as many Points, H or h, may be obtained as you pleaſe; which are all in the Periphery.

That the Point H is in the Periphery is manifeſt. For, ſeeing that the Foci are determined from rhe extremes, C and D, of the Conjugate Axe, making CF and CG each equal to AE, half the Tranſverſe; (by Prob. 1.) conſequently, FH and GH being made alſo equal to AB, the Point H is in the Periphery.

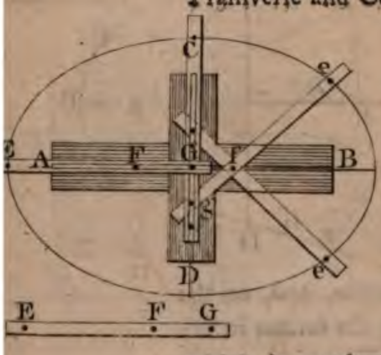
2nd. To deſcribe an Ellipſis, by means of an Inſtrument called a Trammel.

The Trammel is made of two pieces of wood, fixed together at right Angles, in the form of a Croſs; or, for ſmall Ovals, it may be of Metal, with two ſtreight Grooves, truly perpendicular to each other, as AB and CD.

Then, having provided a ſtreight Ruler, of wood or metal, fix a Point or Pencil at one end, as at E; and, let two ſhort, round Pins (equal, in Diameter, to the width of the Grooves in the Trammel, which will ſlide in them, freely) be fixed at F and G, in the manner, or order, as follows.

Make EG equal to half the Tranſverſe, and EF half the Conjugate Diameter.

The Trammel and the Ruler being thus prepared, let the Trammel be placed, as in the Figure; exactly on the Tranſverſe and Conjugate Diameters.



Then, apply the Ruler, with the Pins in the Groove, along the Tranſverſe Diameter, G being in the Center, and the point E in the Periphery, at the extremity of the Tranſverſe.

When the point E is moved towards C, the Pin at G, in the Center, falls into the Groove towards D, whilſt the Pin at F, moves towards the Center; which is continued till it falls into the Center, at G, and the point E being arrived at C, has deſcribed, by its motion, the

the Curve AC, a fourth part of the Periphery; which has all the variety of the whole; for every fourth part is the ſame; from C to B, and from A to D, it is inverted.

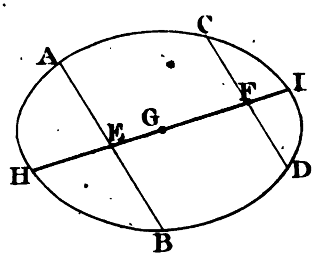
The Ruler being now in the poſition of the Conjugate Diameter; the Pin, at F, in the Center, and that at G fallen down to g; the motion is continued to B; the Pin at F, being now ſuppoſed to be in the Center, moves towards B, and the Pin at g goes back again to the Center; when the Point has deſcribed the Curve CB, another 4th part, the converſe of the former; thus continuing the motion until the Curve is compleated, and the point E arrives again at A; the Pin, at F, ſtill moving, to and again, in the Tranſverſe, from F to f and back again to F; and the Pin at G, to and again, in the Conjugate Groove; by which means a true Ellipſis is formed; which, it is evident, has no part of the Curve of a Circle in its compoſition, ſeeing it is deſcribed on two Centers.

The nearer the Foci are together, that is, the leſs the difference is between the Tranſverſe and Conjugate Diameters, the nearer it approaches to a Circle, and at laſt ends in a Circle, when the Centers unite.

PROBLEM II.

To find the Center of an Ellipſis.

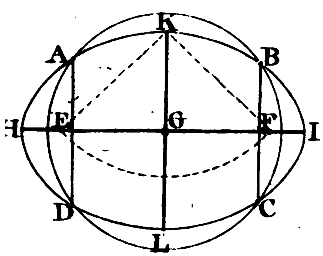
Draw, at pleaſure, two parallel lines, AB and CD, in the Ellipſis. Biſect them, in E and F; through which Points draw HI, which is a Diameter, of the Ellipſis. Biſect the Diameter, HI, in G, which is the Center of the Ellipſis.



For, every Right Line drawn through the Center is a Diameter; and every Diameter is biſected in the Center.

PROBLEM III.

To find the Axes of an Ellipsis.



Having found the Center, by the last, with any radius, less than the Transverse Diameter and greater than the Conjugate, describe a Circle, cuting the Ellipsis in four points, A, B, C, and D.

Draw the Ordinates AD and BC and bisect them, in E and F; through E and F draw HI, which is the Transverse Axe.

Through the Center, G, draw KL, at right Angles with the Transverse, or parallel to the Ordinates, AD and BC, and that is the Conjugate Axe.

PROBLEM IV.

To find the Foci of an Ellipsis.

Having found the Center, G, and drawn the Transverse and Conjugate Diameters, or Axes, take GH, half the Transverse, for radius, and on either extreme, K or L, of the Conjugate, as a Center, describe an Ark, cuting the Transverse in two points, E and F, which are the Foci, or Centers, on which the Ellipsis is described.

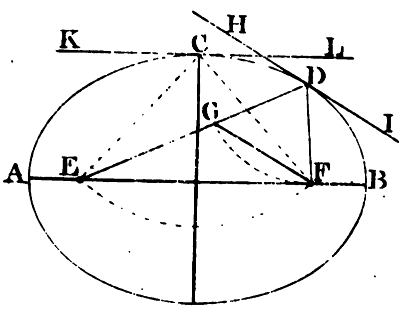
PRO-

PROBLEM V.

To draw a Tangent to an Ellipſis at any Point given; in the Periphery.

If the Tangent was required at either extreme of the tranſverſe or conjugate Axes, as B or C, it will be perpendicular to them, as in the Circle, and conſequently makes equal Angles.

Let D be the given Point, through which a Tangent is required to be drawn.



Find E and F, the Foci of the Ellipſis, by the foregoing; and draw ED and DE.

Make DG equal DF, and draw GF; to which, if HI be drawn parallel, through the given Point D, it will touch or be a Tangent to the Ellipſis in that Point.

For, let KL be drawn, through C, parallel to the Tranſverſe, AB; it will touch the Periphery in C.

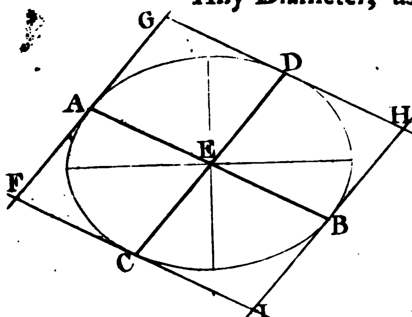
Draw EC & CF; they are equal, by Conſtruction - Pr. 4. and, conſequently, the Angle ECK, is equal LCF - 4. 1. El. for the Angles CEF, CFE are equal - - - 9. 1. El.

But, the Tri. GDF is Iſoſceles, and HI is par. to GE. conſ. the Angle HDG or E is equal to FDI. - 4. 1. El. Therefore, HI touches the Ellipſis, in the Point D.

PRO-

PROBLEM VI.

Any Diameter, as AB, being given, to find its Conjugate, CD.



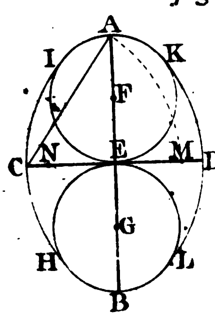
Through either extreme of the Diameter, as A, draw the Tangent FG; through the Center E, draw CD, parallel to the Tangent; which is Conjugate to the Diameter AB.

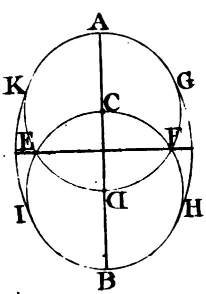
If Tangents are drawn, through the four points A, B, C and D, of any two Conjugate Diameters, meeting in F, G, H and I, the Parallelogram FGHI, is equal to a Rectangle under the two Axes.

PROBLEM VII.

To make a Representation of an Ellipsis, with Compasses.

Let AB be the given Transverse Axe, and CD the Conjugate.





Bisect AE and EB, in F and G, and on the Centers F and G, describe the Circles AIEH, & EKBL, touching at E.

Draw CA; and, with that radius, on C, describe the Ark AM, cutting the Conjugate Axe in M.

Make EN equal EM; on which Centers, M and N, with the radius MC and ND, describe the Arks HI and KL, falling into the Circles AE and EB, at H, I, K, and L.

Or

Or thus, when the difference between the Axes is leſs.

Divide the Tranſverſe, AB, into three equal Parts, at C and D; on which, with the radius AC, deſcribe two Circles, cuting each other in E and F; on which Centers, with the radius AD, deſcribe the Arks GH and IK.

This laſt, is nearer to an Ellipſis than the other; and they may be lengthened or ſhortened at pleaſure. It is very obvious that they are not Ellipſes, but compounded of circular Curves; whereas, a true Ellipſis has no part of the curve of a Circle, in its compoſition, the Curve, being every where deſcribed on two Points, is continually varying; wherefore, no Oval, formed by Circles, can partake of the properties of an Ellipſis.

Having explained or defined all the Terms peculiar to the Ellipſis, and ſhewn how to conſtruct it of any given Dimenſions, and to fix the Points on which it is deſcribed, there remains nothing more to be done in reſpect of the utility of it, to Mechanics, &c.

To treat at large of its Properties would neither be proper nor neceſſary, in this Work, for the reaſon mentioned in the Prelude or Introduction.

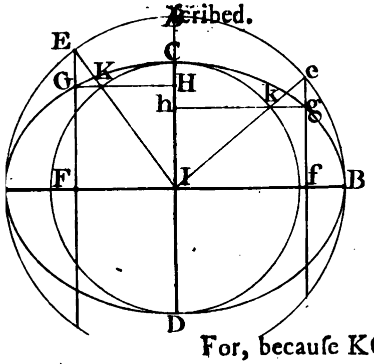
The Properties of the Ellipſis are really very extraordinary and ſurprizing; many of which have ſo near Affinity to thoſe of a Circle, that they are almoſt neceſſarily deduced from them. But, as it is requiſite to have ſome acquaintance with the Conic Sections, in order to a right and clear underſtanding of the Properties of the Ellipſis, which would not be proper to enter upon in this place; I ſhall content myſelf with giving one or two general Theorems, from which, the preceding Problems are deducible; and juſt mention two or three particular Properties, and then proceed to the Elements of Geometry, from which this may properly be called a Digreſſion, though a very uſeful one; and which, I am perſuaded, will not be unacceptable to many.

THEOREM I.

The Rectangle, under any two Abſciſſas, has the ſame Proportion to the Square of the Ordinate which divides them, as the Rectangle under any other two Abſciſſas, has to the Square of the Ordinate dividing them.

In a Circle, the Rectangle, under the two Segments of a Diameter is equal to the Square of an Ordinate, at the point of biſection; conſequently they have all the ſame Ratio (Pr. 14. 3. El.)

Let ACBD be an Ellipſis; and let there be deſcribed two Circles, the one circumſcribing it, the other, fCFD, inſcribed.



From any Point, E, in the outer Circles Periphery, draw the Right Line EF, parallel to the Conjugate Axe, CD, cutting the Periphery, of the Ellipſis, in the Point G; from which Point, draw GH, parallel to the Tranſverſe, AB; alſo draw the Radius, EI, which will cut the inſcribed Circle in the ſame Point K.

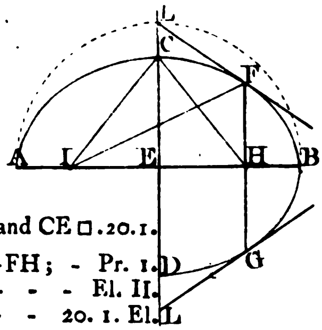
For, becauſe KG is par. to IF, FE:FG::IE:IK, - 2. 6; i. e. as IB:IC. And, EF par. to CD will be cut, in G, by the Periphery of the Ellipſis, as IB to IC. - 19. Em. C. Sec.

Now, GH is parallel to IB, and EF to CI - - - Con.
wh. the Angle HKI=EIF, and HIK=IEF - - 4. 1. El.
conſequently, the Triangles, IHK and IEF, are ſimilar;
wh. as IE:EF::IK:IH. - - - - - - - - - - 4. 6.
But, AI=IE, IC=IK; and FG=IH; - - - - - 15. 1.
wh. AI:EF::IC:FG; conſ. AI:IC::EF:FG, alternately;
and conſequently, AI□:IC□::EF□:FG□. - - - 15. 6.
But, AF×FB is equal to EF□, - - - - - - - - 14. 3.
and, AI×IB is the ſame as AI□; for AI=IB.
Therefore, as AI×IB:IC□::AF×FB:FG□.

The ſame holds true of any Abſciſſas. whatever, and the Ordinate. For, Af×fB:fg□::AI×IB:IC□
conſ. Af×fB:fg□::AF×FB:FG□.

2. The

2. The Latus Rectum (FG) is a third Proportional to the Transverse and Conjugate Axes.



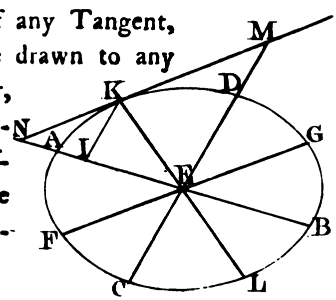
The common method of proving it, is to make FG to AB, as CE $\square$ to AE $\square$; i. e. as CD $\square$ to AB $\square$ (4. 2. El.) which is making FG a third Proportional to AB and CD (12 and 13. 6.) and then, proving it to be what it is by Construction.

The distance of the Focus from the Center = the Sq. Root of the diff. between AE (eq. CH) $\square$ and CE $\square$. 20. 1.

Now, $IF+FH=AB=2EB$, i. e. $IF=2EB-FH$; - Pr. 1.
wh. $IF\square=4EB\square+FH\square-4EB\times FH$. - - - - El. II.
But, $IF\square=IH\square+FH\square$; - - - - - - - - 20. 1. El.
wh. $4EB\square+FH\square-4EB\times FH=IH\square+FH\square$.
conf. $4EB\square-4EB\times FH=IH\square=4EH\square$, - - - Ax. 7.
and, $4EB\square=4EB\times FH+4EH\square$. - - - - - - - El. II.
i. e. $4EB\square=4EB\times FH+4CH\square-4CE\square$.
wh. $4EB\square+4CE\square=4EB\times FH+4CH\square$.
But, $CH=EB$ (Pr. 1.) wh. $4CE\square=4EB\times FH$;
i. e. $4CE\square=2EB\times 2FH$; i. e. $CD\square=AB\times FG$.
conf. FG is a third Proportional to AB and CD. - - 9. 6. El.

3. If from the Center, E, of an Ellipsis, EL (on the conjugate Axe) be made equal to the Semitransverse, AE, the Tangents LF and LG, drawn from that Point, will pass through the Extremes of the Latus Rectum.

4. If from the Point of Coutact, K, of any Tangent, KN, to an Ellipsis, an Ordinate, KI, be drawn to any Diameter, cuting it in I; half that Diameter, AE, is a mean Proportional between the Segment EI, made by the Ordinate, and the Distance of the Point N, from the Center, where the Tangent cuts the Diameter, AB, produced; i. e. EI:EA::EA:EN.



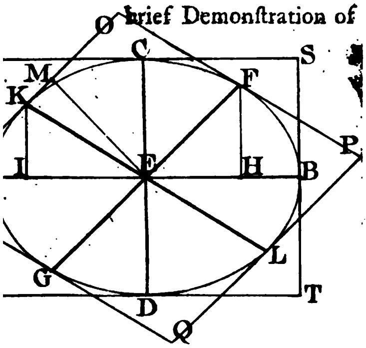
5. If a Tangent MN be drawn through the Extreme of any Diameter, KL, cuting any other two conjugate Diameters, AB and CD, in M and N; the Rectangle under KM and KN, the Segments of that Tangent, will be equal to the square of EF, half the Diameter FG, conjugate to the Diameter KL.

 6. If

6. If from the Extremes, F and K, of any two conjugate Diameters, Ordinates are drawn to the transverse Axe, cuting it in H and I; then is the Ordinate FH to the semi-conjugate Axe, as the Segment EI, made by the other Ordinate, to the semi-transverse Axe,
i. e. FH:CE::EI:EA; and IK:CE::EH:EB.

7. Every Parallelogram formed by Tangents at the Extreme of any two conjugate Diameters, is equal to a Rectangle, under the transverse and conjugate Axes.

As this is a curious and extraordinary Property, I shall give a brief Demonstration of it, as follows.



Let NOPQ be a Parallelogram, made by the Tangents, NO, OP, &c. at the extremes of the Diameters GF and its Conjugate KL.

Draw the Ordinates, FH and KI, to the transverse Axe, cuting it in H and I; and draw EM perpendicular to NO; and consequently,

the Triangles EFH, ENM are similar. - - - - 4. 6. El.
for, the Angles NME, EHF are Right; and MNE=FEH. 4. 1.
Wherefore, - EF:FH::EN:EM. - - - - - - by 4. 6.
Now, - - FH:CE::EI:EA (6.) and EI:EA::EA:EN (4.)
wherefore, - FH:CE::EA:EN; by equality of Ratios.
Then, since, EF:FH::EN:EM; and, FH:CE::EA:EN;
consequently, EF:CE::EA:EM; by inordinate equality;
wherefore, - EF × EM = CE × EA - - - - - by 9. 6. El.
But, - CE × EA = $\frac{1}{4}$ of the Rectangle under AB and CD.
And, - - EF × EM = $\frac{1}{4}$ of the Parallelogram NOPQ. - 18. 1.
Therefore, the Par. NOPQ is equal to the Rect. RSTU. Ax. 5.

ELEMENTS OF GEOMETRY.

BOOK I.

THIS firſt Book of Elements treats of particular properties of Right Lines and right lined Figures, reſpecting themſelves, and alſo their relation to, or connection with one another; of Angles, formed by the interſections of Right Lines, &c. which, being entirely unconnected with Figures, in this Work, are firſt treated on, before we begin with Figures.

Next we proceed with Triangles, the firſt, yet moſt uſeful and extenſive, of all right lined Figures. Then is ſhewn the affinity between Triangles and Parallelograms, and other four ſided Figures; all which, is neceſſary, in ſome degree, to prepare the way for the laſt grand Propoſition, the 47th of Euclid, which concludes this Book.

The Order in which theſe things are treated, according to Euclid, ſeems to me not very regular; the chain of connection is frequently broken or interrupted; beginning firſt with Figures, then Lines and Angles; again Figures and Lines, alternately. The fourth Propoſition begins the properties of Triangles, in general; the 5th introduces the Iſoſceles, a particular ſpecies of Triangles, merely (in this place) for the Demonſtration of the 7th; and that, is of no other uſe but to demonſtrate the 8th, which may be clearly demonſtrated (according to Proclus) without it. In the 24th and

25th is again introduced the ſame general properties of Triangles, which are evidently deducible from the fourth. For, having, in the fourth, proved, that, in two Triangles, if there are found two Sides, in each, reſpectively equal, and that the Angles, contained between the equal Sides, are alſo equal, the remaining Sides are equal; can any Perſon of common ſenſe, who knows what a Plane Angle means, want Demonſtration, that if, in the Caſe of equality of the Angles, the Sides oppoſite thoſe Angles are equal; if one of the Angles be greater or leſs, the ſide oppoſite will alſo be greater or leſs? and, *vice verſa* (in the 25th) if the Side or Baſe, lying between the equal Sides, be greater or leſs, the Angle it ſubtends muſt neceſſarily be ſo too: theſe things ſeem, to me, ſuch a neceſſary conſequence, that I would not heſitate to take them for granted.

This Book is much encumbered with uſeleſs Demonſtrations, of ſeveral Propoſitions, already fully demonſtrated or need none; for, what is evident in itſelf, needs no other Demonſtration; and ſuch Propoſitions, as may eaſily and clearly be deduced from ſome preceding ones, are as well made Corollaries to them; ſuch are all converſe Propoſitions; for I cannot conceive it uſeful to perplex and embarraſs the minds of Youths, with demonſtrating what is not eſſential and abſolutely neceſſary.

I have, therefore, greatly abridged, and altered the Elements of Euclid's firſt Book; yet, I am greatly miſtaken, if I have not retained all that is eſſential in it; and which, I think, may be much eaſier and more regularly acquired. After abſtracting the 14 Problems, I have but 20 other Propoſitions in this firſt Book; three of which number (the 3d the 6th and 16th) are not in Euclid; whereas, excluſive of the Problems, Euclid has 34 Theorems, twice the number, of the remaining ſeventeen.

The

The 7th of Euclid is not only ufelefs, in itfelf, but the Demonftration is intricate, to a beginner; and fince the 8th. which is made to depend on it, is clearly deducible from the 4th, I fee no ufe for it all. In refpect of the 4th (my 8th) I cannot think the Demonftration of it perfectly geometrical; but, fuch as it is, has been generally accepted; and although it has been thought fomewhat deficient, yet, I have not feen even an attempt at any other kind of Demonftration; which is fufficient conviction that no other can be given. As it is the firft Propofition on the properties of Figures, which might well pafs for an Axiom, the Demonftration, being purely mental, is fatisfactory; though it fuppofes a manual, or mechanical application of one to the other; the proof arifing, from which, would be, merely, occular.

Having received the firft rudiments of Geometry from Pardie, I may perhaps have imbibed a prejudice, in refpect of his manner of treating the firft Elements. Upon the whole, I look on that Tract as very imperfect and irregular; but I have always thought it more introductory to treat, firft, of Lines and Angles, before Figures; keeping them diftinct and unconnected, as much as poffible. The 13th. of Euclid I have, therefore, made the firft; for, on the foundation of it, depends, in a great meafure, the whole firft Book. I am the more confirmed in it, having feen (fince my Plan was formed) two or three Authors, of fome Fame, who have purfued the fame Plan, nearly; which at firft (I freely own) hurt me, not a little, but, which, I am now reconciled to, and think it does me credit.

According to Euclid, the properties of parallel Lines &c. cannot be obtained without having recourfe to Triangles, and is the reafon why the 27th, and the following, are not introduced fooner, which are previoufly neceffary to demonftrate the 32nd; in which, the 16th and 17th are more fully and perfectly demonftrated: wherefore, Euclid

was

was obliged to have recourſe to the 16th, as a Lemma, (which is much more complex and difficult to conceive than the 32nd) in demonſtrating ſeveral Propoſitions previous to them. But, for what uſe the 17th is introduced, I am at a loſs to deviſe, it being never once referred to, I believe, in the whole Elements; it might (if it be neceſſary) be a Corollary after the 32nd, and it is never uſed before. We can never be at a loſs to know, that any two Angles of a Triangle are leſs than two Right Angles, having full Demonſtration (in the 32nd) that, the three Angles of every Triangle are, together, equal to two Right ones; conſequently it is of no uſe at all.

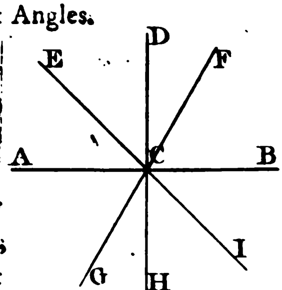
In the 18th of this, is contained the 35, 36, 37 and 38th of Euclid including alſo, in a Corollary, the 39th and 40th. This property of Parallelograms and Triangles is, undoubtedly, very extenſive; but it is all included in the firſt part of the 18th (i. e. in the 35th of Euclid) having previouſly demonſtrated, in the 17th, that every Triangle is equal to half a Parallelogram, having the ſame Baſe and Altitude; which, Euclid deduces and demonſtrates from the other. To dwell ſo long on one Property, in ſix Propoſitions, becomes tedious, if not trifling; for where is the difference, whether Parallelograms, or Triangles, have the ſame or an equal Baſe?

The alterations, which I have made in the firſt Book, are very conſiderable; and the manner of demonſtrating is, in general, different, and more conciſe; as to the clearneſs of it, it muſt be left for others to determine. I do not doubt but the judicious, candid, and impartial Reader will concur with me, or not diſapprove, entirely, the liberty I have taken. I thought it incumbent on me to advertiſe him of the alterations, and, by pointing them out, he may, with more eaſe, form a judgment of them.

THE

THE first six Propositions, and the Corollaries deduced from them, contain all the properties of Right Lines, and Angles formed by their Intersections; which are essentially necessary to be known, before we can obtain a knowledge of the properties of Plane Figures. But, I think it scarce worth the loss of time to demonstrate them, they being in a manner self evident; insomuch, that the greatest part of them may be, and are, by some, given as first Principles, or self evident Propositions.

1. For, since (by Def. 10.) one Right Line standing perpendicularly on another makes the Angles on each side equal to the other, they are, therefore, Right Angles.



The Perpendicular, CD, being common to both Angles, ACD and DCB; the two other Sides AC, CB are, consequently, in one Right Line, ACB. And, it is clearly evident, that any other Line, or number of Lines, as CE, CD, CF, at the same Point C, must make all the Angles, ACE+ECD+DCF+FCB, equal to the two Right Angles (ACD+DCB.)

Hence, the first Proposition and its Corollaries are evidently clear and manifest.

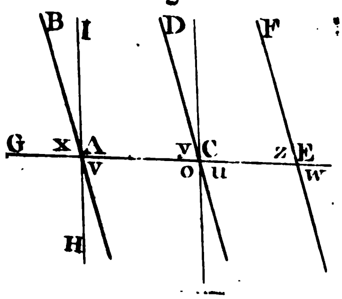
2. Again; (by Def. 11.) if the Perpendicular, CD, be produced, towards H, there is generated equal Angles, to the opposite, on either Side, for, they are all Right Angles, consequently equal, by the 9th Axiom.

So likewise, if any other, inclined, Line be produced, as EC, or FC, it will, necessarily, produce equal Angles, to the opposite, on the other side of AB; as ACG equal FCB, and GCB=ACF; and which, together, are also equal to two Right Angles.

For, if EC biſects the Right Angle ACD, CI will alſo biſect the oppoſite, HCB; and, if CF triſect the Right Angle DCB; the oppoſite Angle, ACH, will alſo be triſected, if FC be produced, towards G; and ſo, in whatever proportion DCB is divided, by FC, the continuation of FC will, neceſſarily, divide the oppoſite, ACH, in the ſame Proportion; wherefore, the Ang. GCH=DCF, and ACG=FCB; alſo, GCH+HCB=DCF+ACD; and ACG+GCB=FCB+ACF; i. e. they are each equal to two Right Angles; and conſequently, all the Angles about the Point C are equal to four Right Angles.

Hence, it is plain, that the firſt and ſecond Propoſitions, with their Corollaries, are clearly deducible from the 10th and 11th Definitions; after what has been ſaid in the Theory of Plane Angles is well underſtood.

3. It is alſo, I preſume, eaſy to be conceived, and which I think no Perſon, at leaſt, who has any Talent for Geometry) can be at a loſs to conceive; that, two or more parallel Lines muſt neceſſarily have the ſame Inclination to any Right Line which cuts them all; i. e. the Angle of Inclination, with each Line, is equal to one another; the bare Idea of Paralleliſm ſeems to indicate it; which being known, or underſtood, all the reſt follow of courſe, viz. that the Alternate Angles are equal to one another, and the two internal Angles, on each Side, are equal to two Right Angles.



For, ſuppoſe the Right Lines AB, CD, and EF to be parallel amongſt themſelves, and all cut by any Right Line, GE; making, with them, the Angles x, y, z; and, ſuppoſe AB to be moved, directly, to CD, ſtill keeping parallel to CD, it muſt neceſſarily coincide with CD; and, conſequently, the

the Angle GAB will always remain the ſame, and, at laſt, coincide with ACD.

And if AB be moved directly forward, to EF, ſtill keeping aarallel to its firſt poſition, it will alſo coincide with EF, and the Angle GAB, or ACD, with CEF.

Hence, it is plain that the Angles x, y and z, made with GE, are all equal to one another; and, being equal to one another, the Right Lines AB, CD and EF are conſequently parallel between themſelves.

4. Again; ſince the Angles x, y and z are equal to one another; and the oppoſite Angles, v, u, w, are alſo equal between themſelves, and to the other (as it was made evident in the foregoing) conſequently, the Alternate Angles, v and y, u and z, &c. are equal; for, they are all equal to one another, as above (the Lines AB, CD and EF being parallel) and, converſely, if the Alternate Angles are found to be equal, the Lines, forming them, are conſequently parallel.

5. Alſo; ſince a Right Line cuting another Right Line makes the Angles, on both Sides, equal to two Right Angles (by the firſt) it is evident, that, if the Angles x, y and z are equal to one another, and the adjoining Angles x+A and y+C, &c. are each equal to two Right Angles; conſequently, A, C, and E are alſo equal between themſelves, and the internal Angles (between each two Lines) A added to y, and C added to z, are alſo, each, equal to two Right Angles; by the 6th. Axiom.

6. And, becauſe A+v and y+o are each equal to two Right Angles; and, A+y, on one Side GE=two Right Angles; conf. v+o, on the other Side GE, are, alſo, equal to two Right Angles. Therefore, if two parallel Right Lines, AB & CD, &c. are cut by any other Right Line, GF, the internal Angles, on each Side, are equal to two Right

Angles; and, conſequently, if a Right Line cuts two Right Lines, which are not parallel (as CD and HI) they will meet on that Side where the two internal Angles (DCE + CEI) on the ſame Side, are leſs than two Right Angles; which, is Euclid's 12th. Axiom.

Now ſince all theſe properties of Right Lines, and of Angles formed by Right Lines, are manifeſt, from what has been advanced already, there is, I think, but little occaſion for Demonſtration; nevertheleſs, in conformity to the Antients, I have formed them into Propoſitions, for the ſatisfaction of the ſcrupulous, and for the ſake of reference, in other Works, as well as in this; yet, I muſt own, I am clearly of opinion, that this brief extract is ſufficient, towards attaining all the neceſſary knowledge inculcated by the firſt ſix Propoſitions. It will, at leaſt, be of uſe to the young Student, in giving him a clear Idea of the Properties therein demonſtrated; which, by being more familiarized to them, will appear more evident and conſpicuous; the neceſſary knowledge, contained in them, will be deeper rooted and more ſecurely eſtabliſhed, and the Pupil better prepared to proceed with the Demonſtrations of the properties of Right lined Figures.

An explanation of the Notes and Abbreviations made uſe of in this and in other mathematical Treatices.

$=$ Note of Equality.
Thus; $A=B$, ſignifies that A and B are equal; and is thus read, A is equal to B; frequently, A equal B.

$+$ Note of Addition, Plus, or more.
Thus $A+B$, ſignifies the Sum of A added to B, A and B repreſenting any Quantity, whatever. Let A be 5, and B, 3; then $A+B=8$.

$-$ Note of Subtraction; Minus, or leſs.
Thus $A-B$, ſignifies the ſum of B taken from A. $A-B=2$, muſt be read, A, leſs B, is equal to 2.

$\times$ Note of Multiplication.
Thus $A\times B$, ſignifies, A multiplied into B. $A\times B=15$. A multiplied into B is equal 15.

$\div$ Note of Diviſion.
Thus, $A\div B$ ſignifies, A divided by B. Let A be 6 and B 2; then, $A\div B=3$.

$:::$ Note of Equality of Proportion.
Thus, $A:B::C:D$, ſignifies, that A bears the ſame Proportion to B, as C to D. $6:9::2:3$.

∺ Note of continued Proportion, or geometrical Progreſſion.
Thus, $A:B:C:D$ ∺ ſignifies, that A has the ſame Proportion to B, as B has to C, and, as C to D. In Numbers, thus, $1:3:9:27$ ∺

$\square$ Square. Thus, $AB\square$, ſignifies, the Square of the Line AB. A——B.


Rectangle. Thus, $AC▭$; or ABC, ſignifies the Rectangle under the Lines AB and BC; or, $AB\times BC$. And, $ABC▭=BD\square$, muſt be read thus; the Rect. ABC is equal to BD Square, or to the Square of BD.

Rect-

Rectangles and other Parallelograms, are frequently denoted by two Letters, at oppoſite Angles. As AC or DB.

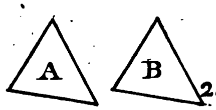
2 ABC; or 2 AB▭; muſt be read, two Rectangles ABC, or twice the Rect. AB; and, 2 or 3 AB □ Square muſt be underſtood, twice or thrice the Square of AB.

△ Triangle. Thus, △ABC, ſignifies the Triangle ABC.

AXIOMS.

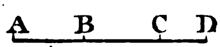
An Axiom (as already defined) is a manifeſt Truth, clear in itſelf, and therefore, does not require to be demonſtrated: Nevertheleſs, an Illuſtration will not, I preſume, be thought ſuperfluous or unneceſſary.

1. Things which correſpond, coincide, or mutually agree with each other, in every Part, are equal.
And if the wholes be equal, the halves are alſo equal.

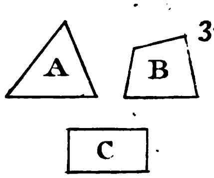


Thus, A is equal B; if every part of the Figure A coincides with B.

2. The whole is greater than a Part of the ſame Thing. For it is equal to all its Parts.

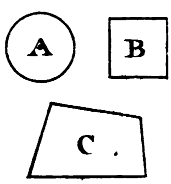


Thus, $AD = AB + BC + CD$.



3. If two, or more, Things, or Quantities, are each equal to the ſame third Thing or Quantity, they are equal between themſelves.

Thus, if $A = C$, and if $B = C$, then $A = B$.

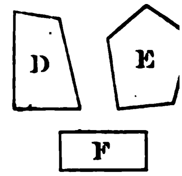


4. Things, or Quantities, which are each equal to half (or any other equal portion) of the ſame third Thing, are equal to one another.

Thus, if A and B are each equal to half of C, A and B are equal.

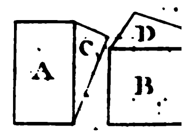
5. Things,

5. Things, or Quantities, which are each double, or triple, &c. of the ſame third Thing, or Quantity, are equal to one another.



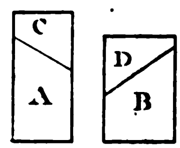
Thus, if D and E are each equal twice or three times F, D and E are equal.

6. If to equal Things be added equals, the Sums are equal.



If to the equal Rectangles A and B, be added equals, C and D; then, $C+A=D+B$.

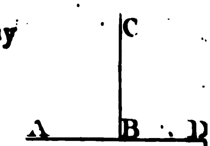
7. If from equal Things be taken away equals, the remainders will be equal.



If from the equal Rectangles, A and B, be taken away equals, C and D, the remainders of A and B are equal.

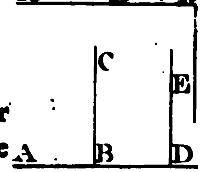
8. If to or from unequal Things be added or taken away equals, the Sums or Remainders will be unequal.

This neceſſarily follows from the two laſt.



9. All Right Angles are equal to one another.

The Right Angle ABC=the Right Angle BDE.



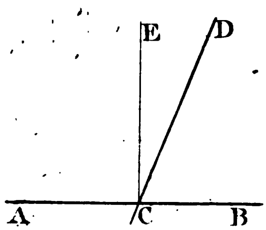
10. Two or more Right Lines, being perpendicular to the ſame Right Line, and being all in the ſame Plane, are parallel. BC is parallel to DE.

The Angles ABC, BDE, being Right Angles, are equal; by the laſt.

THEO-

THEOREM I.

If one Right Line touch or cut another Right Line, at any Point apart from its Extremes, it will make two Angles; which are either both Right Angles, or, both together, equal to two Right Angles.



Let DC be a Right Line, touching AB in the Point C, making two Angles, ACD, DCB; I say, they are either Right Angles, or both together equal to two Right Angles.

If CD be perpendicular to AB the thing is manifest; by Def. 10.

But if CD does not make Right Angles with AB; draw CE perpendicular to AB.

Dem. Now, ACE, ECB are Rt. Angles (Con.) th. equal. But, ACD=ACE+ECD; also DCB=ECB−ECD. Therefore, ECD, the excess of the Obtuse Angle, ACD, to the Right Angle ACE, is also the deficiency of the acute Angle DCB to the Right Angle ECB.

Conf. ACD+DCB is equal to ACE+ECB. - Ax. 2.

Cor. 1. Hence, if from the same Point in a Right Line (as C in the Line CD) two Right Lines AC, CB are drawn, on contrary Sides and in the same Plane, making Right Angles, or Angles equal to two Right Angles, with the given Line, they will constitute one Right Line. ACB.

2. Two Angles at the same Point, and on the same Side of a Right Line, being equal, they are consequently Right ones.

Wherefore, when two Angles are contiguous or adjoinin, if one of them ACE, be a Right Angle, the other ECB is also a Right one; and if one of them ACD be Obtuse, the other, ECB, is consequently Acute.

3. Two

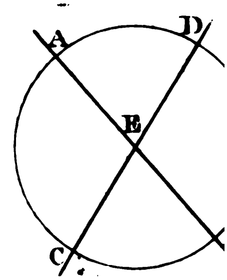
3. Two or more Right Lines meeting in the fame Point of another Right Line, make all the Angles, together, equal to two Rt. Angles. ACE + ECD + DCB = ACE + ECB.

4. When one Right Line makes Angles with another, if one of the Angles be known the other is alfo known.

For, it is the difference between the Angle known and two Right ones. As, ACD = ACE + ECB − DCB.

THEOREM II. 15. Euclid.

Vertical or oppofite Angles, made by the Interfection of two Right Lines, are equal to one another.



Let AB and CD interfect in E; then, the Angles AEC, BED, are equal; alfo, AED is eq. to CEB.

DEM. The Right Line AE, touching the Right Line CD, makes the Angles AEC, AED, together, equal to two Right Angles. - Th. 1.

And, becaufe DE touches the Right Line AB, the Angles AED, DEB, are alfo equal to two Right ones. - - - - - - - - - fame.

Therefore, the Ang. AEC + AED = AED + DEB. - Ax. 9. and if there be taken away the common Angle AED, there will be left, the Ang. AEC equal to DEB. - Ax. 7.

In the fame manner, AED may be proved equal to CEB.

Or, it may be demonftrated after another manner, thus.

About E, the point of Interfection, as a Center, defcribe a Circle (with any Radius at difcretion) cuting the two Lines in A, C, B, and D. Then, the Ark AD + DB = CB + BD.

For, they are, each two, equal to the Semi-circumference of a Circle. - - - - - - - See N. B. Def. 21.

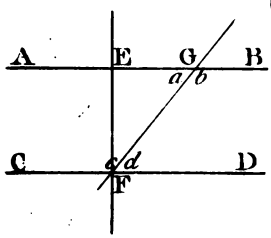
Wh. taking away the Ark DB, there remains AD = CB; and, the Angles AED, CEB, having equal Arks, are equal.

COR. Hence, if two Right Lines intersect each other, the Angles they make are equal to four Right Angles.

2. All the Angles which can be made about a Point are equal to four Right Angles.

THEOREM III.

If a Right Line, cuting two Right Lines, makes all the Angles, on both Sides, equal between themselves, the two Lines are parallel.



Let the Right Line EF cut two Right Lines, AB and CD, making the Angles AEF, EFC, BEF, and EFD equal to one another; then AB is parall. to CD.

DEM. Because the Angle AEF=BEF, EF is perpendicular to AB. - - - - - - - - - - - - - - Def. 10.
And, because EFC=EFD, EF is, also, perpend. to CD.
Wherefore, since EF cuts both Lines, AB and CD, perpendicularly, the Angles AEF, EFC, &c. (being equal) are Right Angles; and, AE, EB, CF, and FD are perpendicular to EF. - - - - - - - - - - Def. 11.
But, AE, EB is the same with AB, & CF, FD, with CD.
Therefore, AB and CD, being both perpendicular to EF, are parallel. - - - - - - - - - - - - Ax. 10.

COR. From hence it is manifest, that, any Right Line, cuting two parallel Lines, makes the internal Angles, on each Side, equal to two Right Angles.

For,

For, if they are cut perpendicularly, as EF, it is clear.

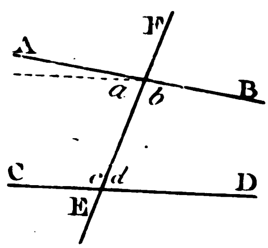
And, ſince any other Line (FG) cuting them both, makes the adjoining Angles (a and b, c and d) with each Line (AB and CD) equal to two Right Angles. - - - Th. 1.
Conſ. all the internal Angles (a, b, c, and d) which it makes with them both, are equal to four Right Angles. - Ax. 2.

Now, ſince AB is parallel CD (Theo.) and the two internal Angles, on one Side EF, are equal to thoſe on the other Side (Hyp.) it is manifeſt, that the internal Angles, on one Side, are, alſo, equal to thoſe on the other Side, of any Right Line (FG) cuting them both; wh. $a+c=b+d$.

But, all the four internal Angles are equal to four Right Angles; conſequently, the internal Angles, $a+c$, and $b+d$, on each Side, are, each two, equal to two Right Angles.

COR. 2. If two Right Lines, which are not parallel, be cut by another Right Line, they will meet on that Side where the internal Angles are leſs than two Right ones.

For, if $b+d$ be leſs than two R. Angles $a+c$ is greater; becauſe, $a+b+c+d=$ four Right Angles. - - Th. 1.
Wherefore, AB will incline to CD, and they will, if produced, meet on that ſide EF, towards B and D, on which the Angles $b+d$ are leſs than two Right ones.

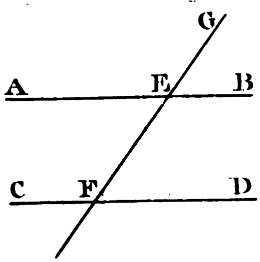


Becauſe, the Angles, b and d, on that Side EF, are, together, leſs than $a+c$, on the other Side.

If this Theorem cannot be admitted, as perfect Demonſtration, it may be ſuppoſed a Lemma to the next; being an illuſtration, at leaſt, of Euclid's 12th Axiom; which, in my opinion cannot be received as ſuch, by any Perſon as yet unacquainted with Geometry. From that conſideration, it may paſs with the moſt ſcrupulous. My deſign is to make Geometry underſtood, and intelligible to common Capacities, in the eaſieſt manner.

THEOREM IV. 27. Euclid.

If a Right Line, cūting two Right Lines, makes the Alternate Angles equal, to one another; or, ſhall make the external Angles equal to the internal and oppoſite; the two Lines are parallel.



Let FG be a Right Line, cuting the two Right Lines AB and CD, and making the Angles AEF, EFD, equal to one another; then, the two Lines, AB and CD, are parallel.

For, if the Angles AEF, EFC, be not equal to two Right Angles, they are either greater or leſs; ſuppoſe them greater; then, the Lines, AB and CD, will meet on the other Side, towards B and D. - - - - - - - - - - - C. 2. 3.

Now, the Angles BEF, EFD, are leſs than AEF + EFC.
And the Ang. AEF + FEB = EFC + EFD, - - Ax. 9.
for, they are, each two, equal to two Right Angles.-Th. 1.
But, the Ang. AEF = EFD (Hyp.) wh. BEF = EFC.-Ax. 7.
Conſequently, BEF + EFD = EFC + EFD;
and, BEF + EFD = AEF + EFC, - - - - - Ax. 6
i. e. they are, on each ſide, equal to two Right Angles.
Therefore, AB is parallel to CD. - - - Th. 3. and Cor.

2. Let the Angle GEB be equal to EFD.

Now, the Ang. EFD = GEB (Hyp.) and, AEF = GEB. 2.
Wherefore, AEF is equal to EFD. - - - - Ax. 3.
But, the Angles AEF, EFD, are Alternate; and, it is already proved, that, if Alternate Angles be equal the Lines are parallel. Therefore AB is parallel to CD.

COR.

Cor. 1. Conv. If a Right Line cuts two Right Lines, which are parallel, the Alternate Angles are equal to one another.

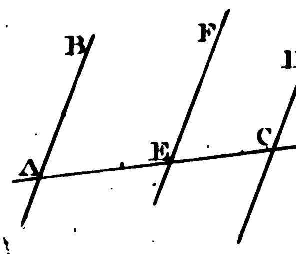
For, AEF+BEF are equal to two Right Angles. Th. 1.
Alſo, BEF+EFD are equal to two Rt. Angles. C. 1. 3.
Wherefore, AEF=EFD (BEF being common) Ax. 3.
and conſeqnently, BEF=EFC; by the ſame.

Cor. 2. If a Right Line cuts two parallel Lines, it makes equal Angles with them both. viz. GEB equal to GFD.

N. B. From this Theorem, is deduced the fifth Problem; according to the firſt Method, there preſcribed.

THEOREM V. 30. Euclid.

If two, or more, Right Lines are parallel to any other Right Line, they are parallel between themſelves.



Let the two Right Lines, AB and CD, be both parallel to the ſame Right Line, EF.

I ſay, that AB is parallel to CD.

Let any Right Line, (AG) cut them all; being produced, if neceſſary

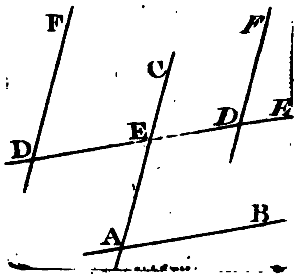
Dem. Becauſe AB is parallel to EF, and they are both cut by the Rt. Line AG, the Angle BAE=FEC. - - C. 2. 4.
And, for the ſame reaſon, the Angle DCG=FEC;
wherefore, the Angle BAE is equal to DCG. - Ax. 3.
But, thoſe Angles are internal and oppoſite. - Def. 16.
therefore, AB is parallel to CD. - - - 2. Th. 4.

SCHOL. *This Theorem might well paſs for an Axiom; for, by adopting parallelïſm for Things, or Quantity, 'tis the ſame as Axiom the third.*

THEO-

THEOREM VI.

If two Right Lines cut each other, and, if there be two other Right Lines (in the ſame Plane with the former) parallel to them, reſpectively, and alſo cut each other, they contain equal Angles.



Let the Right Lines DE and DF (or *DF*) be reſpectively parallel to AB and AC.

I ſay, the Angle FDE is equal to CAB.

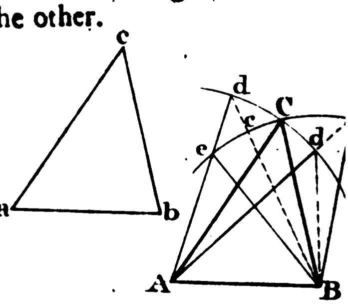
Produce either Side (if it be neceſſary) as DE (or *ED*) cuting AC, at E.

DEM. Now, becauſe DE is parallel to AB, and they are both cut by AC, the Angle CAB=CE*D*. - - C. 2. 4.
And, becauſe DF (or *DF*) is parallel to AC, and are both cut by DE, the Angle FDE=CE*D*. - - - - ſame.
But, the Angle CAB=CE*D*; and FDE=CE*D*. proved.
Therefore, the Angle FDE is equal to CAB. - - Ax. 3.

This valuable Theorem is not in Euclid, except the 10th of the eleventh Book, may be compared with it; in which, the two Lines (cuting each other) are, reſpectively, in different Planes; but they are, neceſſarily, either in the ſame Plane, or in parallel Planes, and holds equally true.

THEOREM VII. 8. Euclid.

If the three Sides of one Triangle are equal, respectively, to the three Sides of another Triangle, they are alſo equiangular; i. e. the Angles oppoſite to equal Sides, are alſo equal, and the whole Triangles are congruous, or equal, each to the other.



Let the Triangles a b c, ABC, have all their three Sides equal; viz. a b equal to AB, b c equal BC, and a c equal AC.

I ſay, the Angle a is alſo equal to A, b equal B, and c equal C; and the Triangle, a b c, equal to ABC.

DEM. Suppoſe the Triangle a b c applied to ABC; any Side, a b, to its equal AB. - Poſt. 5.

Then, if the Triangles are alike ſituated, a c will fall on AC, and b c on BC; the Angle a will coincide with A, b with B, and c with C.

For, becauſe the Side a b is equal to AB, the Extremes, a and b, will fall on A and B; and, becauſe a c is equal to AC, and b c to BC, the Point c muſt neceſſarily fall on C

If not, let c, if poſſible, fall at d, in the Ark C d, deſcribed on A, with the radius a c; wh. A d = AC, equal a c.

But, B d is not equal to b c; for, if an Ark of a Circle be deſcribed, on B, with the Radius b c, being equal to BC, it will cut the Ark, d d, at C, and the Line A d, at e. Wherefore, the Point c cannot fall at d, nor elſewhere but on C, in the interſecting Point of the two Arks.

Th. the Triangles, agreeing in every part, are congruous.

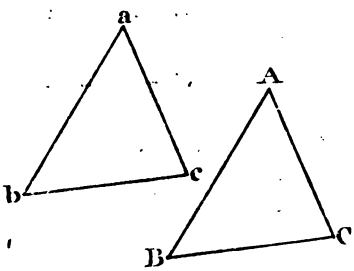
N. B. This Property does not hold good in any other Figure.

Hence, an Angle is made equal to a given Angle, as in Pr. 4; and hence, an Angle is biſected according to the 2nd method, Prob. 9th; viz. by making two Triangles, congruous with each other.

THEO-

THEOREM VIII. 4. Euclid.

Triangles having two Sides, in each, respectively, equal, and if the Angles, contained by those Sides, are also equal; then, the remaining Side, of the one, shall be equal to the remaining Side of the other; the remaining Angles, of the one, equal to the remaining Angles of the other, viz. those that are opposite equal Sides; and the whole Triangles are equal, and congruous.



Let the Triangle, a b c, have two Sides, a b, b c, equal, respectively, to AB, and BC, of the Triangle ABC; and, let the Angle a b c be equal to ABC.

Then, the Side a c is equal to AC; the Angle a is equal A, and c equal C; and the whole Triangle, a b c, equal to ABC.

DEM. For, suppose the Triangle, a b c, applied to, or laid upon, the Triangle ABC, in such wise, that the Side a b, of the one, agrees with AB of the other Triangle; the Extreme a, with A, and b with B. - - Post. 5.

Then, the Side b c will fall upon BC; - by Def. 14; for, they contain equal Angles, by the Hypothesis.

The Extreme c, of the one, will coincide with C, of the other; because, b c is equal to BC.

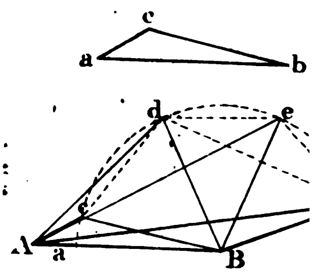
Therefore, the remaining Sides, a c and AC, must necessarily agree, and be equal to each other; the Angle a will coincide with A, and c with C; and consequently, the whole Triangles, a b c, ABC, are congruous; therefore the Triangle a b c is equal to ABC.

COR.

COR. Hence, if two Sides of one Triangle be equal, respectively, to two Sides of another Triangle, and the contained Angle, of one, be greater or less than the contained Angle of the other, the remaining Side will also be greater or less; and conversely.

For, suppose two Triangles, a b c, ABC, having their Sides, a b and b c, respectively equal, to AB and BC.

It is evident, that, if the Angle a b c was the least possible; i. e. if the inclination of the Sides, a b, b c, was such, that it is not possible to conceive a less Angle, the two Sides, a b, b c, would nearly coincide; and consequently, the length of the remaining Side, a c, would be somewhat more than the difference between a b and b c; viz. a c=a b−b c, nearly.



Again. Suppose the Angle made by the Sides AB, BC, equal a b, b c, the largest possible, so as, nearly to fall into one Right Line; it is manifest, that the remaining Side, AC, would be somewhat less than the length of both, AB and BC; i. e. AC=AB+BC, nearly.

Hence it is plain, that, whether the Angle, a b c or ABC, contained between a b and b c, or AB and BC, be greater or less, the remaining Side, a c or AC, will also be greater or less; and conversely, the greater or less the Side, a c or AC, the greater or less is the Angle, a b c or ABC.

For, suppose the Triangle a b c applied to ABC, the Side a b to AB, the Angle a to A, and b to B; and let the Ark of a Circle be described, from c to C, on the Center B.

Then, suppose the Angle a B c opened or enlarged, to the several Apertures, a B d, a B e, a B C; it is evident, that as the Angle a B c is increased, at a B d, &c. so is the remaining Side, A d, A e, &c. continually longer; from A c, the shortest possible, to AC, the longest.

Hence the 24th and 25th of Euclid are clear, and manifest.

S What

What I have here advanced is, I presume, sufficient conviction, if not Demonstration, of the properties therein contained.

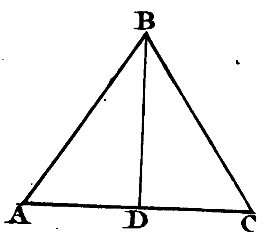
The shortest way to knowledge is the best; where perfect Demonstration is necessary, I shall be as brief as is consistent with perspicuity, and where it is not, I shall briefly illustrate.

Those who are not satisfied with this Demonstration of the two last, self evident, Propositions (of Euclid) may refer back, after the 12th (and this is not of use between) and, by drawing the Right Lines d e, e C, &c. they will find, that the Side A e, or AC opposite to an obtuse Angle, A de, or A e C, is greater than the Side A d, or A e, which is opposite to an acute Angle, A e d, or AC e.

The 7th might be a Corollary to the 8th. For since, in two Triangles, if there be found two Sides, in one, equal respectively to two Sides in the other, and containing equal Angles, the third Sides are equal; it certainly follows, that if all the three Sides are respectively equal, the Triangles are equal and congruous; seeing it is not possible for an Angle to be greater or less, but the Side opposite is also greater or less.

THEOREM IX. 5. Euclid.

In Isosceles Triangles, the Angles at the Base are equal to one another, i. e. if a Triangle have two equal Sides, the Angles, opposite to them, are also equal.



Let ABC be an Isosceles Triangle; the Side AB equal to BC.

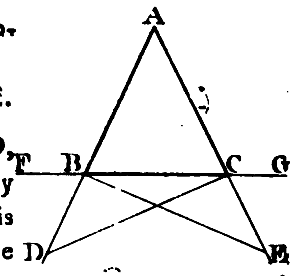
Then, the Angle A is equal to C.

Bisect the Angle ABC (Pr. 9.) made by the equal Legs, AB, BC, and draw BD it will also bisect the Base, AC.

Dem. In the Triangles ABD, DBC, the Sides AB, B
are equal (Hyp.) and, BD is common to them both.

Now, the Sides AB, BD are = CB, BD, respectivel
And the Angle ABD is equal to DBC. - - - Co
Th. AD=DC, and the Triangle ABD=DBC. - 8
But, in congruous Triangles, opposite equal Sides are
equal Angles. - - - - - - - - - Th. 7 and 8.
Th. the Ang. A=C, being opposite the common Side BD.

Or it may be demonſtrated thus, after Euclid.



Let the equal Sides, AB, AC, be produced below the Baſe, to D and E. Make AD eq. AE, and join DC and BE.

Then, in the Triangles ABE, ACD, there are two Sides, AB, AE, reſpectively equal to AC, AD; and the Angle A is common; wherefore, DC=BE; and the Angle ABE is equal to ACD. - - 8.

But, BD=CE (Ax. 7.) and BC being common; the Triangle, and conſequently, the Angle, DBC=BCE. - 7. Alſo, the Angle BCD=CBE; conſ. ABC=ACB. - Ax. 3.

Cor. 1. The external Angles, made by producing equal Sides of a Triangle, or the Side which lies between them, are alſo equal, amongſt themſelves.

For, the Ang. DBC=BCE; conſ. ABF=ACG. - 2.

2. An Equilateral Triangle, is alſo equiangular.

For, oppoſite to equal Sides are equal Angles.

3. If a Triangle have two equal Angles, the Sides oppoſite them are alſo equal. And, if all the three Angles of a Triangle are equal, it is equilateral.

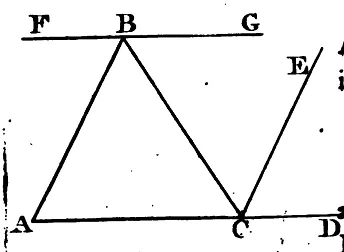
4. A Right Line, biſecting the Angle, and the Baſe, contained between equal Sides, is perpendicular to the Baſe; and conſequently, two Right Lines drawn from the ſame Point in a Perpendicular to a Right Line, to Points, equally diſtant from the Perpendicular, in that Line, are equal.

For, the Angles ADB, BDC, being equal, are conſequently Right Angles. (See firſt Figure.) C. 2. Th. 1.

N. B. Hence are performed the 6th, 7th and 8th Problems; viz. by conſtructing Iſoſceles Triangles.

THEOREM X. 32. Euclid.

If any Side of a Triangle be produced, the external Angle is equal to the two remote Angles of the Triangle. And, the three Angles of every Triangle are, together, equal to two Right Angles.



In the Triangle ABC, let any Side, as AC, be produced. Then, the Ang. BCD is equal to the two Angles, A and B.

From C, draw CE parallel to AB.

Becaufe CE is parallel to AB - Con. and they are both cut by the Right Line BC, the Angle BCE=ABC. - Th. 4.
And, becaufe AB and CE, being parallel, are cut by the Right Line AD, the external Angle, ECD=BAC, the internal and oppofite; - - - - - - - - 2. Th. 4.
But, the Ang. BCE (eq. B)+ECD (eq. A)=BCD. Ax.-2.
Therefore, the external Angle, BCD,=A+B, the two remote Angles of the Triangles. Q. E. D. - Ax. 3.

2. The three Angles, A+B+BCA, are equal to two Right Angles.

Having produced AC, and drawn CE, parallel to AB. -

DEM. The Angle BCE=ABC, and ECD=BAC - 4.
Wh. BCD (eq. BCE+ECD) is equal to ABC+BAC.
But, the Angle ACB,+BCD= two Rt. Angles. - - 1.
Therefore, the three Angles, A+B+BCA, are equal to two Right Angles.

This 2nd part is alfo elegantly proved, by drawing a Right Line, FG, through any Angle, B, parallel to the oppofite Side, AC.

DEM.

DEM. For, the Angle FBA $=$ A, and GBC $=$ BCA - 4.
conſ. the Ang. A $+$ ABC $+$ BCA (eq. FBA $+$ ABC $+$ CBG) are equal to two Right Angles. - - - C. 3. Th. 1.

COR. 1. Hence, if one Angle of a Triangle be equal to the other two Angles, it is a Right one.

For, if either Side, containing the Right Angle, be produced; the external Angle, being equal to the two remote ones, is equal to the Right Angle to which it is contiguous. (Defin. 11.)

2. There can be but one Right Angle, or one Obtuſe Angle, in a Triangle.

Becauſe, if there be one Right Angle, the other two are, together, equal to a Right Angle.

3. If a Triangle have one Right Angle, the other two are acute, and equal to a right one; and, if the Triangle be Iſoſceles, the two Angles, at the Hypothenuſe, are each equal half a Right Angle. For, they are equal. - Th. 9.

4. If the ſum of two Angles of a Triangle be known, the other is alſo known.

For, it is the Complement of two Right Angles.

Alſo, if one Angle be known, the Sum of the other two is known.

5. If the Sum of two Angles of a Triangle, either together or ſeparately, be equal to two Angles of another Triangle, the two remaining Angles are equal.

For, the Sum of all the Angles of every Triangle is the ſame; i. e. they are equal to two Right Angles.

6. The equal Angles of every Iſoſceles Triangle are acute.

For, however ſmall the Angle at the Vertex, the Angles at the Baſe are, each, equal to half the difference between that Angle and two Right Angles.

7. The

7. The Angles of an Equilateral Triangle are each 60 Degs.
For, it is two thirds of a Right Angle, or one third of two Right Angles.

8. All the Angles of every Quadrilateral are, together, equal to four Right Angles.
For, by drawing a Diagonal, it is divided into two Triangles; and all the Angles, of each Triangle, are equal to two Right Angles; confequently, the fum of all the Angles, of both, are equal to four Right Angles; and they are equal to all the Angles of the Quadrilateral.

Note. The two following Theorems are alfo deducible from the laft Propofition; which, not being elementary, are not numbered amongft the other Theorems.

THEO. 1. All the internal Angles, of every right lined Figure, are equal to twice as many Right Angles, wanting four, as the Figure has Sides.

Every right lined Figure, may be refolved into as many Triangles, as the Figure has Sides, wanting two; by drawing Diagonals, from any Angle, to all the oppofite Angles.

In the irregular Hexagon ABCDEF, let there be drawn the Diagonals AC, AD, and AE; which divides it into four Triangles, ABC, ACD, &c.

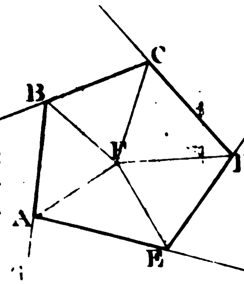
DEM. Becaufe, by the Theorem, the three Angles of every Triangle are equal to two Right Angles, the Sum of all the Angles of the four Triangles, ABC, ACD, &c. are equal to eight Right Angles.

And, they are equal to all the internal Angles of the Hexagon. Therefore, all the Angles of the Hexagon are equal to eight Right Angles.

But, a Hexagon has fix Sides, and twice fix is twelve; by taking away four, there will remain eight Right Angles; which are equal to all the Angles of the Figure,

Or;

Or; by aſſuming any Point, as F, within the Figure, ABCDE, and drawing Lines to every Angle, from that Point, FA, FB, &c. it will be divided into as many Triangles, as the Figure has Sides.



Now, the Angles of all the five Triangles are equal to ten Right Angles. - - Theo. 10.

But, one Angle, of each Triangle, is at the point F, within the Figure; and all the Angles about a Point are equal to four Right Angles (Cor. 2. 2.) conſequently, the remaining Angles, of the Triangles, are equal to all the Angles of the Figure; which, in a Pentagon, are equal to ſix Right Angles.

Cor. Hence, the Sums of all the Angles, of every right lined Figure, having an equal number of Sides, are equal.

Theo. 2. All the external Angles, made by producing every Side, of any right lined Figure, are equal to four Right Angles.

For, ſince the internal and contiguous external Angle, BAE+BAG, &c. are equal to two Right Angles (Th. 1.) the Sum of all the internal and external Angles, together, are equal to twice as many Right Angles, as the Figure has Sides.

But, all the internal Angles are equal to as many, wanting four; conſequently, the external Angles are equal to four.

Cor. The ſums of all the external Angles (taken together) of every Right lined Figure, are equal.

SCHOL. *This property of right lined Figures is, at firſt, really ſurprizing; for, the external Angles of a Triangle, are equal to all the external Angles of any Poligon, whatever, and of any number of Sides.*

The general utility of the knowledge of this property of Triangles, in the laſt Theorem, is very extenſive. It is, perhaps, the moſt perfect Demonſtration that can be given; and we may conceive an Idea of its extenſiveneſs, from the Corollaries and Theorems already deduced from it.

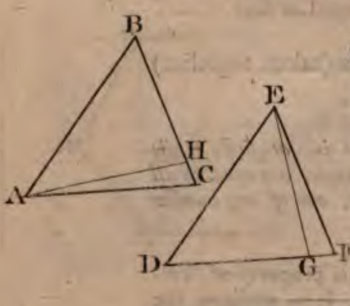
THEO-

THEOREM XI. 26. Eucli

Triangles having two Angles, and one Side, refpe tively equal to two Angles, and a correfpondi Side; whether it be the Side, between the equ Angles, or either of the other, which are oppofi to equal Angles; the remaining Sides and Ang fhall be equal, and the Triangles are congruous.

The firft part of this Propofition, viz. when the equal Sides between the equal Angles, is fo very obvious, that it might ha been made a Corollary to the 8th, from which it is eafily deduc

Dem. For, if AB, equal DE, be applied to DE (Poft. 5 if the Angle A be equal D, and B equal E (the equ Sides, AB, DE, lying between) then will AC fall DF, and BC on EF. - - - - - - - Def. 1 Conf. the Angle C, will fall on F, and the Triang ABC, coincide, or agree in every refpect, with DEF.



2. Let ABC and DEF be two Triangl whofe Angles A and C, of the on are, feparately, equal to D and F, the other; and let the Side AB, of t one, be equal to DE, of the other.

Then, the Triangles, ABC, DE are equal, and congruous.

Dem. Firft, if the Side AC be not equal to DF, it is eith greater or lefs; fuppofe it lefs, and take DG equal A and draw EG.

The

Then, becaufe AB=DE (Hyp.) and DG=AC, by Suppofition; the Sides, AB, AC, are refpectively equal to DE, DG; and the Angle BAC=EDG. - - Hyp.
Wh. EG=BC and the Triangle DEG=ABC - - - 8.
confequently, the remaining Angles, of the one, are equal to the remaining Angles of the other, which are oppofite equal Sides; wherefore, the Angle DGE is equal ACB.
And, DFE=ACB (Hyp.) wh. DGE=DFE. - Ax. 3.

But, DGE is greater than DFE; - - - Th. 10.
for it is an external Angle of the Triangle FEG;
it is, therefore, both equal and greater, which is abfurd.
Therefore, AC is not lefs than DF.

After the fame manner, it may be proved not greater; confequently, AC is equal to DF.
Th. the Triangle ABC is equal to DEF; and congruous.

Again; if it be affirmed, that BC may be either greater or lefs than EF; fuppofe BC greater than EF.
Take BH equal EF, and join AH.

Then, the Angle BAC, being equal to EDF, is greater than the Angle BAH; which deftroys the Hypothefis.
Wherefore, BC cannot be greater or lefs than EF.

This Theorem concludes the general properties of Triangles, mentioned in the preamble to this Book; which I have kept as near together as poffible; by which properties, peculiar to Triangles, only, we can, with certainty, affirm them to be congruous, confequently equal.

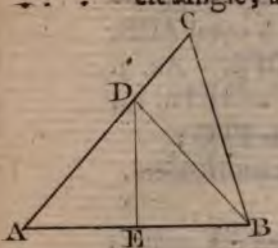
Firft, when all the three Sides, of one Triangle, are equal, refpectively, to the three Sides of another Triangle. (Th. 7.)

2nd. When there are found only two Sides refpectively equal to two Sides, and containing equal Angles between thofe Sides. (8.)

3dly. When two Angles, of the one, are refpectively equal to two Angles of the other; and one Side of each Triangle, either lying between the equal Angles, or oppofite to equal Angles, alfo equal; by this Theorem.

THEOREM XII. 18. Euclid.

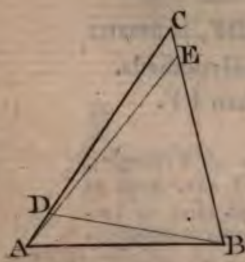
In every Triangle, the greateſt Side ſubtends the greateſt Angle; and the leaſt Side ſubtends the leaſt Angle.



Let ABC be a Triangle, whoſe Side AC, is greater than CB; I ſay, the Angle B, is greater than the Angle A.

Biſect the Side AB, in E; draw ED perpendicular to AB, and join DB.

DEM. In the Triangle ADB, the Side AD=DB; - C. 2, 9.
wherefore, the Angle BAD is equal ABD. - - Th. 9.
But, the Ang. ABD (eq. BAD) is leſs than ABC. Def. 14.
Therefore, the Angle ABC (oppoſite to AC) is greater than BAC, which is oppoſite to a leſs Side, BC.



Or thus, after Euclid.

By Hypotheſis, AC is greater than CB.
Make CD equal to CB, and join DB.

Then, the Ang. CBD=CDB. - Th. 9
But the Ang. CDB is greater than CAB. 10
And, CBD (eq. CDB) is leſs than ABC
Th. the Angle ABC is greater than CAB.

2nd. The leaſt Side ſubtends the leaſt Angle.

AB is leſs than BC; then, the Angle C is leſs than A. Make BE equal BA, and draw AE.

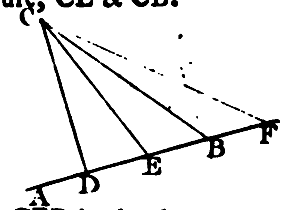
DEM. Now, the Angle AEB=BAE. - - - Th. 9
But, AEB is greater than ACB. - - - - - - - 10
Th. BAC, which is greater than BAE, is greater than ACB, which is leſs than AEB (equal EAB) Q.E.D.

COR-

COR. 1. In Triangles, oppoſite the greateſt Angle is the greateſt Side; and the leaſt Side is oppoſite the leaſt Angle.

COR. 2. A Perpendicular is the ſhorteſt Line, which can be drawn from any Point to a Right Line. And that Line, which falls neareſt to the Perpendicular, is ſhorter than the more remote ones.

Let CD be perpend. to AB. Draw, at pleaſure, CE & CB.

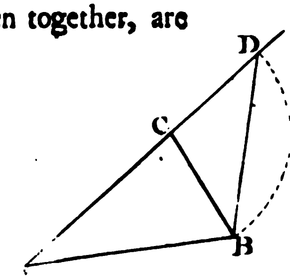


Now, ſince CDE is a Right Angle, CED is acute. - - - - C. 3. 10. Therefore, CE, which ſubtends the Right Angle, is greater than CD. - - Theo.

2ndly. Becauſe the Angle CED is acute, CEB is obtuſe. Conſequently, CB is greater than CE. - - - Theo. Therefore, CE, which is neareſt to the Perpendicular, CD, is leſs than CB, which is more remote from it.

THEOREM XIII. 20. Euclid.

In every Triangle, any two Sides, taken together, are greater than the remaining Side.



Let AB be the greateſt Side of the Triangle ABC.

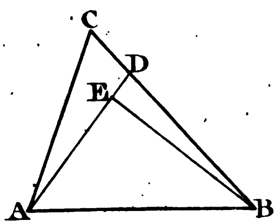
I ſay, the Sides AC, CB, are, together, greater than AB,

Produce AC to D; make CD equal CB, and join BD.

DEM. Then, becauſe CD = CB, the Ang. CBD = CDB - 9. But, the Angle ABD is greater than CBD, equal CDB. Then, ſince in the Triangle ADB, the Angle ABD is greater than ADB; the Side AD is greater than AB - 12. But AD = AC + CB (Con.) therefore, AC + CB, is greater than AB. Q. E. D.

THEOREM XIV. 21. Euclid.

If, from any Point within a Triangle, two Right Lines are drawn to the extremes of any Side; those Lines, taken together, are less than the other two Sides of the Triangle, but they contain a greater Angle.



In the Triangle ABC, assume any Point, E, and draw EA, EB.

I say, that AE added to EB, is less than AC added to CB.

And, the Angle AEB is greater than ACB.

Produce AE to D.

Dem. In the Triangle ACD, the Sides, AC, CD, are greater than AD. - - - - - - - - - - - - 13.

Add DB to both; and AC+CD+DB, (eq. AC+CB) is greater than AD+DB. - - - - - - - Ax. 8.

Again. In the Triangle EDB, the Sides, ED, DB, are greater than EB.

Add AE to both; and DB+AD is greater than EB+AE.

But, AC+CB is greater than AD+DB, - - proved.

Therefore, AC+CB is still greater than AE+EB.

2ndly. The Angle ADB is greater than ACB, and AEB is greater than ADB; - - - - - - - - 1. Th. 10.

consequently, AEB is still greater than ACB.

For, the Angle ADB is external, in respect of the Triangle ACD; and AEB, in respect of BED.

Cor. Hence; if two Triangles have one Side, in each, equal to one another, or have one Side common; and, if the remaining Sides, of the one, be less, respectively, than the remaining Sides of the other, they will contain a greater Angle.

THEO.

THEOREM XV. 34. Euclid.

In every Parallelogram, the oppoſite Sides and Angles are equal; and, it is cut into two equal Triangles by a Diagonal.

Let ABCD be a Parallelogram.

The Side AB is equal to DC, and AD is equal to BC.

Alſo, the Angle D is equal to B, and A equal to C.

Draw either Diagonal; as AC.

DEM. By Def. 33d. AB is parallel to DC; and they are both cut by AC; wh. the Angle BAC=ACD. - Th. 4. And, becauſe AD is parallel to BC, and are both cut by AC; the Angle DAC=ACB; and AC is common; wherefore, the Triangles, ABC, ADC, are congruous - 11. Therefore, the Side AB=DC, and AD=BC. - Def. 43. Alſo, the Angle B=D (Th. 11.) &, DAB=BCD - Ax. 6. (For, BAC=ACD, and DAC=ACB (Th. 4.) as above; and conſequently, BAC+CAD=ACD+ACB.)

COR. Hence, it is manifeſt, if Right Lines be drawn between the extreme Points of two Right Lines, which are equal and parallel (ſo as not to croſs each other) they are alſo equal and parallel,

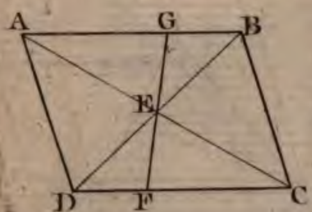
As AD and BC, joining the Points A & D, B & C.

Hence, the 33d of Euclid is evident.

THEO-

THEOREM XVI.

The two Diagonals of a Parallelogram bisect each other. And, every Diameter divides a Parallelogram into two equal and congruous Figures.



In the Parallelogram ABCD, draw the Diagonals, AC and BD.

I say the Diagonal AC is bisected by DB; and, DB is also bisected by AC, in the Point E, of their mutual Intersection.

DEM. Because, AB is parallel to CD, and are both cut by AC and BD; the Ang. BAC=ACD, & ABD=BDC. 4.
But, AB=DC (15.); wh. the Triangle AEB=DEC; and also congruous. - - - - - Th. 11.
wh. DE=EB & AE=EC, being opposite equal Angles.
Therefore, the Diagonals, AC, & BD, are bisected, in E.

2ndly. The Diameter, FG, divides the Parallelogram ABCD, into equal and congruous Figures.
AGFD is equal to FGBC.

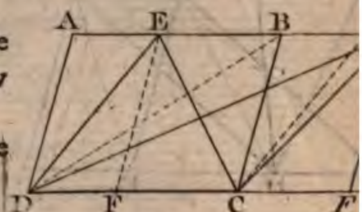
DEM. Now, the Triangle ABC is equal to ACD. - 15.
The Angle AEG=FEC (2.) and GAE=ECF, - 4.
and AE=EC (above) wh. the Tri. AGE=EFC. - 11.
Let them be taken away; there remains AEFD=CEGB.
and conf. △AGE+AEFD=△EFC+CEGB. - Ax. 6.
Th. the Trapezium AGFD is equal to FGBC. Q. E. D.

COR. Every Right Line, which divides a Parallelogram, and passes through the Center, is bisected at the Center.
As, FG is bisected in E.

THEO.

THEOREM XVII. 41. Euclid.

If a Triangle have the ſame Baſe as a Parallelogram and lie between the ſame Parallels, i. e. having the ſame Altitude, the Triangle is equal to half that Parallelogram.



In the Parallelogram ABCD, aſſume any Point, E, in the Side AB, and draw EC and ED:

The Triangle DEC is equal half the Parallelogram, ABCD.

Draw EF, parallel to AD and BC.

DEM. The Parallelograms, AEFD and FEBC, are biſected, by the Diags DE & EC; ADE=DEF, & FEC=EBC-15.
Wh. the Triangle DEF+FEC=ADE+ECB. - Ax. 6.
But, the Triangle DEC=DEF+FEC. - - Ax. 2.
Therefore, △ DEC=ADE+ECB. Q. E. D.

Or; produce AB, make B*E* equal to AE, and draw C*E*.

Then, △ B*E*C (equal AED. 8.) compleats the Par. DE*E*C (15. Cor.) which is equally divided by the Diag. EC. The Triangle DEC=E*E*C (equal AED+ECB) - - 15.

Caſe the Second.

When *E* falls without the Par. ABCD, in AB produced. The Triangle D*E*C is equal half the Par. ABCD.

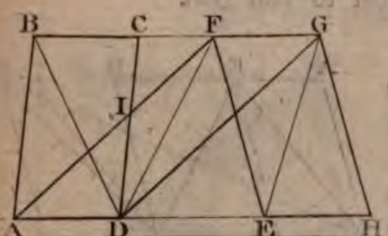
Draw *EF*, as before, parallel to BC; meeting the Baſe, DC, produced, in *F*

DEM. D*E* biſects the Par. A*EF*D; wh. A*E*D=D*EF*-15. and, the Diag. C*E*, cuts the Par. CB*EF*, in BC*E*=*E*C*F*. Wherefore, the Triangle D*EF*—C*EF* (equal D*E*C) is equal half the Parallelogram A*F*—B*F* (equal ABCD.) i. e. the Tri. D*E*C is equal half the Par. ABCD. Q.E.D.

THEOREM XVIII.

Containing the 35th, 36th, 37th, and 38th of Euclid.

Parallelograms, or Triangles, having the ſame or equal Baſes, and equal Altitudes, are equal.



The Parallelogram ABCD is equal to the Parallelogram AFGD; ſtanding on the ſame Baſe A D, and between the ſame Parallels AH, BG; i. e. they have the ſame or equal Altitudes.

Alſo, the Par. ABCD is equal to EFGH, having equal Baſes, AD=EH.

DEM. Now, AB=CD, and AF=DG. - - Th. 15.
And, the Angle BAF=CDG. - - - - - 6.
Wherefore, the Triangle ABF=DCG. - - - 8.
But, the Triangle ICF is common to both;
wherefore, the Trapezium ABCI=IFGD. - Ax. 7.
Add, to both, the Triangle AID; which is common;
the Parallelogram ABCD is equal to AFGD - - Ax. 6.

Again. The Par. ABCD, is proved equal to AFGD.
For the ſame reaſon, EFGH=AFGD, FG being common.
Therefore, the Par. ABCD=EFGH. - - - Ax. 3.
But, AD=FG and EH=FG (15); wh. AD=EH.-Ax.3.
Th. the Parallelograms, AC and FH, have equal Baſes.

Secondly. Draw the Diagonals BD, DF and EG.

Now, ſince every Triangle is equal to half a Parallelogram having the ſame Baſe and Altitude, - - - - Th. 17.
the Tri. ABD=half the Par. ABCD; &, AFD=half AFGD
Therefore, the Triangle ABD is equal to AFD. - - Ax. 4.

And, for the ſame reaſon, the Triangle ABD=EGH.
(For, the Parallelogram ABCD is equal to EFGH.)
And, their Baſes, AD, EH, are alſo equal. Q. E. D.

COR.

COR. Equal Parallelograms or Triangles, having the ſame or equal Baſes, have the ſame Altitude; or, are contained between the ſame Parallels.

This Corollary is the converſe of the Theorem, and contains the 39th and 40th Propoſitions of Euclid.

N. B. From this Theorem, the 24th, 26th, and 27th Problems are deduced; which may be found particularly uſeful to Surveyors of Land, &c.; ſeeing, by its means, any Plot or Survey, though ever ſo irregular, in Figure, may be reduced, with the greateſt facility, and accuracy, to a Trapezium or Triangle; and, its Area obtained at one Operation. (See Prob. 24.)

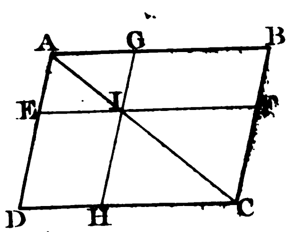
Alſo, by its means, almoſt any right-lined Figure, as well as a Trapezium, may be divided into two equal Parts, by a Right Line from any determined Point, in any Side.

An Example, of which, will be found in the Appendix.

SCHOL. *Hence it is evident, that two Spaces may differ greatly in Circumference, yet contain the ſame Space; and alſo, that two Figures or Spaces, of equal Circuit, may contain very different Areas.*

THEOREM XIX. 43. Euclid.

In every Parallelogram, the Complements are equal.



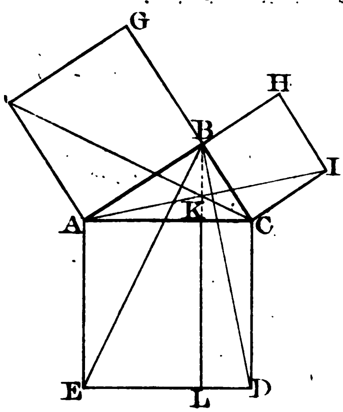
In the Parallelogram ABCD; aſſume any point, as I, in the Diameter AC; through which, draw EF and GH, parallel to AB and AD.

I ſay, the Complements, DI and IB, are equal to each other.

DEM. AC divides the Par. DB into two Tris. ABC=ACD. And the Parallelograms AGIE and IFCH, are divided into equal Triangles AGI=AIE, and IFC=ICH. - Th. 15. Now, if from the equal Triangles ABC, ACD, there be taken away the equal Triangles AGI, IFC=AIE, ICH, there will remain, the Par. DEIH equal to IGBF. - Ax. 7.

THEOREM XX. 47. Euclid.

In every Right-angled Triangle, the Square of the Hypothenuse, is equal to the Squares, of the two Sides, containing the Right Angle.



Let ABC be a Right-angled Triangle.

On the Hypothenuse AC, describe the Square ACDE; and, on the Sides AB, CD, describe the Squares AFGB and BHIC.

I say, the Square AD, of the Hyp. AC is equal to the two Squares, AG and BI, of the Sides, AB and BC.

From the Right Angle, B, draw BK perpendicular to AC; which, produce to L, parallel to CD. The Square AG is equal to the Rectangle AL; and, the Square BI is equal to the Rectangle KD. Draw FC & BE.

DEM. In the Triangles FCA, ABE, AF, AC are respectively equal to AB, AE; and they contain equal Angles. For, FAB=EAC (Ax. 9.) and, if BAC be added to both, the Angle FAC=BAE. - - - - - - - - - Ax. 6.
Wh. FC=BE, and the Triangle FCA=ABE. - Th. 8.
But, the Tri. FCA is equal to half the Square FB - 17.
(for, they have the same Base, FA; and, they are between the same Parallels, GC and FA.)
And, the Tri. ABE (eq. FCA)=half the Rect. AKLE (on the same Base, AE, & between the Parallels BL & AF
Therefore, since the Tri. FCA=ABE (and they a each equal to half the Square AG, or the Rectangle A the Square AFGB is equal to the Rectangle AKLE - Ax

Agai

Again; the Square BI is equal to the Rectangle CKLD;

Draw AI and BD.

Then, in the Triangles AIC, CBD, the Sides IC, CA, are equal to BC, CD, respectively.

And, the Angle ICA=BCD (ACB being common)-Ax.6.

Wherefore, the Triangle AIC=CBD. - - - Th. 8.

But, the Triangle IAC,= half the Square CH; - 17.
(having the same Base, CI, and between the Pars.AH,CI)

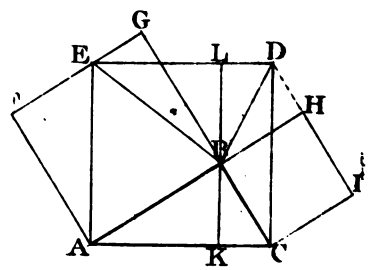
And the Tri. CBD (eq. IAC)= half the Rect. KD - 17.
(being on the same Base, DC, & between the Pars.LB,DC)

Wh. the Square BI is equal to the Rectangle KD-Ax.5.

But, the Square AG was proved equal to the Rect. AL. and AL+KD=AD (the Square of the Hyp. AB.)-Ax.2.

Therefore, the Square AFGB+BHIC=ACDE. Q.E D.

Or, it may be, as elegantly and more briefly, demonstrated after the following manner.



Having described the Squares FB and BI, of the two Sides AB, BC, let the Square of AC, the Hypothenuse, be inverted; as AEDC.

Through B, draw KL parallel to CD; Join BD & BE; and produce IH to D.

Then, the Square AG is equal to the Rectangle AL; and the Square CH is equal to the Rectangle CL.

DEM. For, the Triangle AEB=half the Square AFGB -17.
having the same Base, AB, and Altitude, BG.

The Tri. AEB is also equal half the Rect.AELK - same
having the same Base, AE, and Altitude, AK.

Th. the Square AG is equal to the Rect. AL. - - Ax. 5.

Again; the Tri. BDC=half the Square BHIC; - 17
(on the fame Bafe, BC, & between the Parallels DI, BC)
And the Tri. BDC is alfo equal half the Rect. KLDC
(having the fame Bafe, CD, and Altitude, LD.)
Wh. the Square BHIC = the Rect. KLDC. - Ax.
But, the Rect. AL+LC= the Square AEDC. - Ax. 2
Th. the Square AEDC (of the Hyp. AC) is equal to th
Squares, AFGB and BHIC, of the two Sides, AB & BC

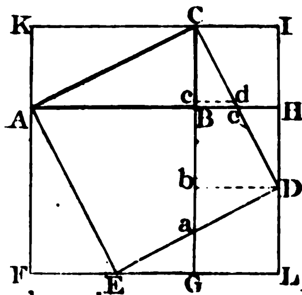
Cor. Hence; if, in a Triangle, the Square of one Side b
equal to the Squares of the other two Sides, the Angl
contained by thofe two, is a Right Angle.

I have been more particular in the Demonftration of this f
mous Theorem, than any of the preceding ones; not becaufe
is the laft of this Book, but, becaufe it is fo extraordinary in itfe
and of fuch fingular ufe throughout the Mathematics. It is al
of great ufe in the following Books of the Elements; infomuc
that, without it, we fhould not be able to advance much furthe
Pythagoras is faid to be the inventor of it; for joy of which,
is faid, that he offered a hundred Oxen to the Deity, who inf
red him with fo noble, and fo ufeful, a Theorem.

That there may not remain the leaft doubt, in the mind of an
concerning the truth it contains, I have added another Figur
the Demonftration of which is both ocular and mathematical.

EX. 3. Let the Square AEDC, of the Hypothenufe A
be inverted, as before; and, having drawn the Square
the two Sides, AB, and BC; produce FA and FG,
and IH, meeting in K and L; forming the Square FJ

The two Rectangles ABCK, and BGLH, are e
being under equal Lines, AB=BG, and BC=BH.
And the four Triangles ACK, AFE, ELD, and
are equal between themfelves, and alfo to the Rect
BL, BK; each being equal to half a Rectangle;
as ACK, equal half ABCK. - - - - -



But, the large Square FLIK, is equal to the two Squares FB, and BI, added to the two Rectangles, BL and BK. - Ax. 2.

It is alfo equal to the Square AEDC, added to the four Triangles ACK, AFE, ELD, and DIC.

And, the four Triangles are equal to the two Rectangles, BL and BK.

Confequently, the Square AEDC is equal to the two Squares, FB and BI, of the Sides, AB and BC.

Note. If the Squares, AFGB and BHIC, are cut according to this Figure, by inverting the Square AEDC; the Parts, fo divided, will exactly cover the Square AEDC.

For, the Perpendicular Db being drawn, and bc made equal to bD (equal CI) draw cd parallel to bD.

Then, the Trapezium, AEaB, is common to both; the Tri. AFE is congruous with ABC, & EaG with Cdc; alfo, the Trap. CIHe will agree with Dbcd; and, CeB will coincide with Dab, which completes the Figure.

Therefore, the Square AFGB+BHIC=AEDC. - Ax. 2.

Having, in thefe twenty Theorems, gone through the firft Book of the Elements of Geometry, (feventeen of which, contain the whole thirty-four, according to Euclid) it will not, I prefume, be impertinent, before we proceed further, to fay fomething of its utility. It is reafonable for a perfon, who has gone fo far, to afk, of what ufe are all thofe properties of Triangles and Parallelograms, &c.? which queftion, though pertinent enough, it may not be fo eafy to give an anfwer to, which fhall be fatisfactory; unlefs we are acquainted with almoft every branch of the Mathematics.

If the ftudy of Geometry was of no other ufe, than for the enuring us to Demonftration; by which we may have conviction in the queft of Truth, it would be great. Geometry is Truth itfelf; the manner of reafoning geometrically, is folid and convincing. It is neceffary for a Perfon who would be a great Orator, at the Bar, &c. to ftudy Geometry; it teaches how to range our Ideas, to ftate the Premifes fairly, and to draw Conclufions from them with clearnefs and certainty. But, when we confider that it is an

an Introduction to the Mathematics, a Key to let in the first Ideas, to further and more extensive knowledge, we shall pursue it with avidity.

The knowledge of the properties of Triangles and Parallelograms is very extensive, and of great use in almost every mathematical science; as Surveying, Navigation, Perspective, Astronomy, and numerous others, depend on them. The Demonstrations of the most simple general properties of Triangles, in Theo. 7, 8, 9, 11, 12, 13, and 14th, are necessary for the Demonstration of others, as well as for reference, for the proof of what is asserted in other Works. The 10th, a general property, and the 20th, a particular one, are more sublime properties of Triangles; and are, perhaps, the most perfect Demonstrations that can be given; which are frequently quoted and referred to hereafter.

The affinity between Triangles and Parallelograms, in Theorems 15 and 17th, is the foundation of Mensuration of Superficies: The properties of Parallelograms and Triangles in the 18th, are very extensive; the equality of the Complements of Parallelograms, in the 19th, is also of great utility; both, which, carry with them the strongest conviction imaginable.

In respect of the properties of Right Lines, and Angles formed by their Intersections, in the first six Theorems, the knowledge of which, though in a manner self-evident, is absolutely necessary for attaining the properties of Figures. I shall just give one instance, as a specimen, of what great things may be effected, from the knowledge inculcated by the fourth Theorem; viz. that the alternate Angles, formed by a Right Line cuting two parallel Lines, are equal.

It would hardly gain credit, with some, to assert, that it is possible from the measure of a small portion of the Earth's Circumference to determine the whole, by means of that simple property; yet nothing is more certain, and that, without much more previous knowledge, than, that the same portion or ark of every Circle, gives or subtends equal Angles at the Center; and consequently, equal Angles being formed at the Centers of Circles of any Radius, will cut off equal portions of the Circumference: (See the Theory of Plane Angles.)

It is very practicable to measure the distance between two places, on a level Plane; and, by means of an Instrument, contrived for that purpose, any space or distance between two places, very remote, may be ascertained, let the ground, between, be ever so irregular, interspersed with Hills and Vallies; in which case, it must be observed, that the measure, so obtained, is not the real Distance between those places, which can only be measured by a Right Line (see Def. 3, and 4.) and which is impossible to be had by an Instrument. And, let the Ground between two remote places be ever so apparently level, it is not a Plane, but a portion of a Sphere;

Sphere; and consequently, the measure, by an Instrument, over the Sruface, is the measure of an Ark of a Circle, of which the Chord Line is the true Distance. (See Def. 22.)

This convexity of the Earth would be obvious, to ocular conviction, if its Surface was regular. In a perfect Calm, when the Surface of the Sea is at rest, it is discernable, that a Ship at a distance, in going from Land, will gradually appear to descend; the Bottom or Hull of the Vessel will soon be lost to sight; and, by means of Telescopes, the Sails and Rigging of the Ship may be seen, when the Hull is apparently sunk in the Water; which alone, is sufficient conviction, to any thinking Person, of the Earth's rotundity.

Now, although it is possible to measure a small portion of the Earth's Circumference, with tolerable accuracy, it would be impossible to measure the whole; because it is, in no part of it, entirely circumscribed with Land; and there is no dependence on the measures taken at Sea; therefore it is impracticable, for several reasons, to obtain its true Circumference by actual measurement.

It is easy to conceive, that two Right Lines or Poles erected perpendicular to the Earth's Surface, at any tolerable distance from each other, are not parallel between themselves; because, the Earth, being a Globe, or globular, every Right Line or Plane, which is perpendicular to its Surface, would, if produced, pass through its Center. Consequently any two Right Lines, which are perpendicular, would, if produced, form an Angle at the Center; which Angle, is greater or less in proportion to the distance of the Perpendiculars from each other (see Art. 1. Theory of Plane Angles.) Hence it is plain, from what has been said, that nothing more is required, than to determine the Angle which two Perpendiculars would form at the Earth's Center, according to the Distance between them.

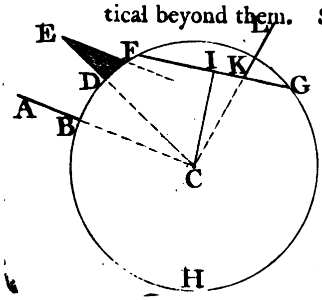
Suppose a small portion of the Circumference (which may be represented by the Circle F G H) be measured, from B to D, where the Earth is as nearly on a level as possible. (By a small portion, I would not be understood to mean less than two or three Degrees, or about 200 miles.)

Let AB represent a Pole, or Stile, erected perpendicular to the Horizon, at B; it will consequently, tend to the Center, at C. Now the Ark BD being measured, due North or South, and another Pole, D E, erected perpendicular to the Horizon, at D, it will also tend to the Center, C, making an Angle, ACE, with AB. The whole mystery is to determine that Angle.

From the immense distance and magnitude of the Sun, in respect of the Earth, its Rays, and consequently the Shadows of Objects, on the Earth, are projected in Right Lines, parallel amongst themselves.

It

It muſt alſo be obſerved, that this operation can only be pe-
formed at or between the Tropics; becauſe, the Sun is never ve-
tical beyond them. Suppoſe the Perpendicular AB to be erect-



When it is in the Meridian of that pla-
the Shadow of AB will be projected
wards C, the Center of the Earth, for
Shadow of it can be projected on the Su-
face, the Sun being perpendicular over i
At the ſame Time, any other perpendicu-la
Line, as ED, at a great diſtance from AB
will caſt a Shadow, as DF, in proporti-on
to the length of the Perpendicular DE.

Now, becauſe ED is perpendicular to FD the Angle EDF
a Right one. (Def. 11.) Wherefore, the meaſure of ED bei-
known, and the length of the Shadow DF taken, DF being co-
ſidered as a Tangent, and DE the Radius, the Angle DEF is d-
termined, by Trigonometry; or, by a Scale of equal parts, ma-
ing a right-angled Triangle of the two given Lines, ED and D-
(Prob. 13.) which will give the Angle DEF.

But, EF is parallel to AB; and EDC is a Right Li-
cuting them both; wherefore, the Angle ACE is equal to CE-
(Th. 4.). Therefore, the Angle, at C is determined.

Now the Angle BCD being known, which, ſuppoſe equal t-
and a half or three Degrees; then, as ſo many Degrees is to 36-
ſo is the Ark BD in Miles, &c. to the whole Circumference; by
the Rule of Three, or Proportion.

Or, having deſcribed a Circle of any Radius, and made an
Angle BCD, equal to DEF, which a Ray of Light makes with
the Perpendicular DE (Prob. 4.) it will intercept a portion of the
Circumference, BD, which being taken in your Compaſſes (beginning at B or D) ſee how oft it is contained in the Circumference FGH; which, multiplied into the number of Miles, meaſured on the Ark BD, will give the meaſure of the Earth's Circumference, in Miles.

Hence it is alſo evident, that, if a large portion of the Earth's Surface was a Plane, inſtead of being ſpherical (of which let FG be a Section) we ſhould, at any diſtance from I, where a Right Line, CI, from the Center, is perpendicular to FG, appear as if on a declining Plane; and the farther from the point I, the greater will be the inclination; inſomuch, that in going from I, towards F or G, we ſhould ſeem as if climbing an aſcent; making, at K, the acute Angle LKG, with the Plane FG. For, a Perſon, ſtanding upright at K, would tend towards the Center C, in the direction LC; and conſequently, would be more inclined to FG, the farther the point K is from the Perpendicular CI.

ELE-

ELEMENTS OF GEOMETRY.

BOOK II.

THE ſecond Book of Elements treats of the properties or powers of Right Lines, divided at pleaſure, and otherwiſe. It generally ſeems, at firſt, ſomewhat ſtrange and obſcure to beginners, who are unacquainted with Menſuration; but, if they carefully obſerve the inſtructions, I have given, concerning Rectangles, all difficulty will ſoon vaniſh. The method or manner of Demonſtration, is, in the firſt eight Theorems, both ocular and mathematical; it is alſo, in the ten firſt, numerical, and does not admit of the leaſt doubt to any, who are tolerably verſed in the fundamental rules of Arithmetic.

It is of eſpecial uſe in attaining a knowledge of the properties of the Circle, as contained in the third Book. For which reaſon, I adviſe the young Student, not to paſs it over ſlightly and with indifference, but to digeſt it carefully, and be ſure he underſtands it perfectly; the ten firſt particularly, before he advances further; elſe, he will frequently be obliged to turn back to them. Therefore, I adviſe him to get them by Heart, ſo, as to be clearly certain of the power of a Line, ſo and ſo divided, whenever it occurs in the courſe of this Treatiſe; it will greatly facilitate his purſuit of the remainder, and ſave him ſome trouble.

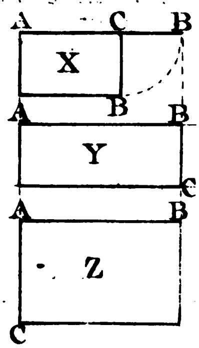
This ſecond Book treats alſo, more fully, of the Theory of Triangles, in general; ſhewing, as in the right-angled, the proportion which the Square of any Side of a Triangle, of any ſpecies, has to the Squares of the other two; by which

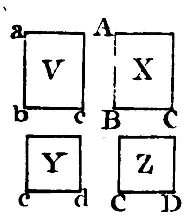
means, the Perpendicular, and, conſequently, the Area of any Triangle may be obtained by calculation, in Numbers, by the meaſures of the Sides only, the Perpendiculars being otherwiſe unattainable. Likewiſe ſome other properties of Triangles and Parallelograms, which are not in Euclid.

I have made no alterations in this Book, nor abridgements; but have added ſome very elegant Theorems. The Demonſtrations, in general, as well in the other Books as this, I have made as brief as is conſiſtent with perſpicuity, and the nature of Demonſtration.

RECTANGLES and SQUARES are already defined. (See Def. 34, & 35, in the general Introduction to Geometry.)

A Rectangle, or right-angled Parallelogram, is ſaid to be under, or contained under two Lines, which are the meaſures of its Sides.





If a Right Line be any how divided, as AB, in C, the Rectangle A C B or B C A is the ſame; and ſignifies a Rectangle under the two Segments, AC & CB; i. e. A C×CB; as X.

But the Rects. A B C and BAC, are different from that, and from each other, if the Line be divided unequally.

ABC ▭ ſignifies a Rectangle under the whole line, AB, and the Segment, B C, or AB×BC; as Y.

And, BAC means one under the whole line, AB, and the other Segment, A C, or BA×AC; as Z.

The middle letter is always twice meant.

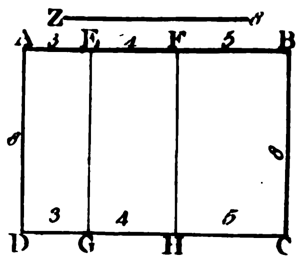
AXIOM. Rectangles or Squares, contained under equal Right Lines, are equal.

As V and X, under the Line a b, eq. AB, and b c, eq BC. And the Squares Y and Z, under c d, equal C D.

THEOREM I.

The Rectangle contained under any two Right Lines, is equal to all the Rectangles, under either Line, and the several Segments of the other, divided into any number of parts, at pleasure.

Let AB and Z be the two given Lines; and let AB be divided, at pleasure, in E and F.



I say, that $Z \times AE + Z \times EF + Z \times FB$ $= AB$ multiplied into Z.

On the Line AB, construct the Rectangle ABCD, whose Sides AD and BC are each equal to Z. - - - - Pr. 18. Through E and F, draw EG and FH, parallel to AD and BC; dividing the Rect. ABCD into three Rect. AG, EH & FC.

Dem. Now, because AD and BC are both equal to Z;-Con EG and FH are also equal Z. - - - Th. 15. 1. & Ax. 3.
For, ABCD is a Rectangle, by Construction;
wherefore, AG, EH and FC are Rectangles.
But, the Rect. AC = all the Rects. AG, EH & FC - Ax. 2.
and the Sides AD, EG, FH and BC, are each equal Z.
Therefore, $Z \times AE + Z \times EF + Z \times FB = Z \times AB$.

In Numbers thus.

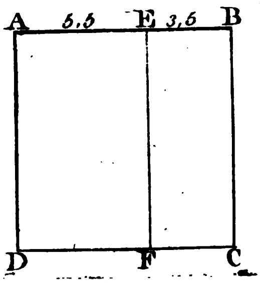
Let AB represent, by some Seale, 12 Feet, or any other measure; and, let Z be equal 8 of those Parts.
Let the Segment, AE be 3, EF 4, and FB 5.

Then; the Rect. AG, i. e. $Z \times AE$, or $8 \times 3, = 24$.
$+$ the Rect. EH, i. e. $Z \times EF$, or $8 \times 4, = 32$.
$+$ the Rect. FC, i. e. $Z \times FB$, or $8 \times 5, = 40$.
$=$ the Rect. AC, i. e. $Z \times AB$, or $8 \times 12 = 96$.

COR. If the two Lines are divided into Parts, at pleasure; the Rectangle under the two whole Lines, is equal to all the Rectangles under the several segments of both Lines.

THEOREM II.

If a Right Line be divided, any how, into two parts; the Rectangles under the whole Line and each Segment, are equal to the Square of the whole Line.



Let A B be divided, at pleasure, in E. I say, that $AB \times AE + AB \times EB = AB \times AB$; i. e. the Square of AB.

On the given Line, AB, describe the Square ABCD; and, through E, draw EF, parallel to AE.

DEM. Now, ABCD is a Square; by Construction; wherefore, AD, and BC are, each, equal to AB - Def. 35.
And AEFD, EBCF, are Rectangles. (Con.)
wherefore EF is equal to AD, or BC; - - - - - - 15. 1.
But, the Square ABCD = the Rectangles AF, FB - Ax. 3.
Therefore, $AB \times AE + AB \times EB = AB \times AD$, or AB □.

In Numbers, let the whole Line, AB, be 9; and let it be divided in AE 5,5, and EB 3,5.

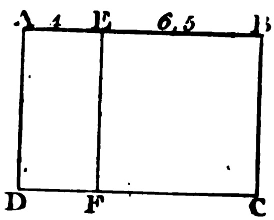
Now, the Rect. AF, i. e. $AB \times AE$, or $9 \times 5{,}5, = 49{,}5$.
$+$ the Rect. FB, i. e. $AB \times EB$, or $9 \times 3{,}5, = \underline{31{,}5}$.
$=$ the Square AC, i. e. $AB \times AB$, or 9×9, $= 81{,}0$.

N. B. This Theorem is only a particular case of the former, viz. when the two given Lines are equal; and might have been a Corollary to the first; but I was not willing to break the order of Euclid unnecessarily.

THEO-

THEOREM III.

If a Right Line be divided at pleasure; a Rectangle under the whole Line and either of the Segments, is equal to a Rectangle under the two Segments, added to the Square of the Segment, first taken.



Let AB be divided in E.

I say; that AB×BE=AE×EB+EB □.

Construct the Rectangle ABCD, under the whole Line AB and the Segment EB; and draw EF parallel to AD.

Dem. The Rectangle AEFD added to the Square EBCF is equal to the Rectangle ABCD. - - - - Ax. 2.
But, the Rect. AF is under the Segments AE, EB, (eq. EF) and, FB is the Square of EB, the Segment first taken.
Th. AB×EB (or ABE ▭)=AE×EB (AEB ▭)+EB □
Also, AB×AE=AEB ▭+AE □, of the lesser Segment.

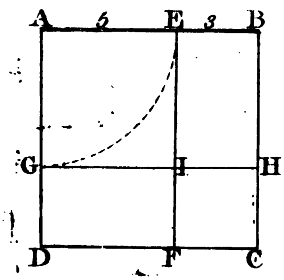
Let the whole Line, AB be 10,5; let AE be 4, & EB, 6,5.
Then, the Rect. AF, i. e. AE×EB, or 4×6,5=26
+ the Square FB, i. e. EB×EB, or 6,5×6,5=42,25
= the Rect. ABCD, i. e. AB×EB, or 10,5×6,5=68,25

N. B. I have given this Line, in Numbers, fractional; and also divided the former, fractional-wise, in order to shew, that it will hold true in Numbers (as in Lines) any how divided; which, I advise the young practitioner to try, by various fractional divisions, if he be conversant in Decimals; it will be of some service, and make him more perfect in the whole Theory of this second Book.

THEO-

THEOREM IV.

If a Right Line be divided at pleaſure; the Squares of the two Segments, added to two Rectangles under the Segments, are equal to the Square of the whole Line.



Let A B be the given Line; divided, at pleaſure, in E.

Then will the two Squares of AE & EB added to two Rectangles under AE & EB be equal to the Square of AB.

On AB, conſtruct the Square ABCD and draw EF parallel to AD.

Make AG equal AE, and draw GH parallel to AB.

DEM. Then, becauſe ABCD is a Square (Con.) the Sides AD, DC, & BC, are each equal, to AB; and the Angle A, B, C, and D are all Right ones. - - - Def. 35

Wh. ſince AG was made equal to AE, GD is equal to EB

But, GH is parallel to AB, and EF is parallel to AD; wh. HC, and FI, IH, and FC, are each equal to EB.

EI and GI are, for the ſame reaſon, each equal to AE.

And, the Angles AEI, IHC; AGI, and IFC, are eac equal to ABC, and ADC. - - - - 2. Th. 4. 1

Wherefore, AEIG and IHCF are Squares of AE and EB and GF, EH are Rectangles under the Segments AE & EB

And, they are all equal to ABCD, the Square of AB-Ax.

Therefore AE□ + EB□ + 2 AE × EB = AB□.

Let the Line AB be 8, divided at E, in 5 and 3.

Then,	the Square G E,	i. e. A E □,	or	5×5,=25
+	the Square F H,	i. e. E B □;	or	3×3,= 9
+	two Rect. GF & EA.	2 AEB▭,		2.5×3,=30
=	the Square ABCD,	i. e. A B □,	or	8×8,=64

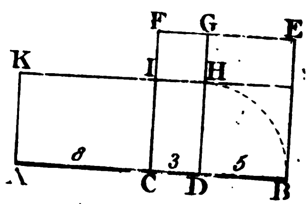
COR. If a Line be equally divided, the Rectangle under the Segments is a Square, and is equal to the Square of either Segment.

Hence, the Square of a whole Line is equal to four times the Square of half the Line. $4 \times 4 \times 4 = 8 \times 8, = 64$.

THEOREM V.

If a Right Line be bisected, and also cut unequally at pleasure; the Rectangle, under the two unequal parts added to the Square of the intermediate part, i. e. the difference between the equal and unequal, is equal to the Square of half the Line.

Let AB be cut equally in C, and unequally in D. Then, the Rectangle under AD and DB, added to the Square of CD, is equal to the Square of AC or CB.



On CB, half AB, describe the □ CBEF. Draw DG parallel to CF; make DH equal DB, and draw KH parallel to AB, and AK parallel to CF.

DEM. Now CFEB is a Square (Con.) and AC=CB - Hyp.
wh. BE=AC (Ax. 3.) & AK, CI, are each =DH-Def. 7.
But DH=DB (Con.) wh. AK & CI are each =DB-Ax. 3.
Th. the Rect. AKIC is equal to the Rect. DGEB. - Ax.
Add, to both, the Rect. FD; &, AI+FD=FD+GB - 6.
but AI+ID, eq. AH, =ADB▭, & FGHI is the □ of CD; for, DG is parallel to CF, and are each equal to CB -Con. also, KH is parallel to AB, and CI is equal to DB; consequently, IF and GH are each equal to CD. - Ax. 7.
Th. the Rect. AH + the Square FH=the Square CFEB. - 2.
i. e. AD×DB+CD □=AC, or CB □.

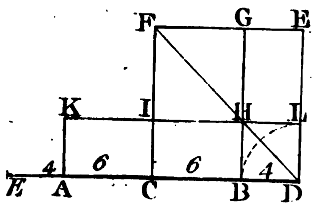
Let

Let the whole Line AB be 16; AC=CB each =8; let CD be 3, and DB 5; then AC+CD, =AD, =11.

Then, the Rect. AH, i. e. ADB ▭, or $11\times5=55$
+ the Square FH, i. e. CD □, or $3\times3=\ 9$
= the Square CFEB, i. e. CB □, or $8\times8=64$

THEOREM VI.

If a Right Line be divided into two equal Parts, and then produced at pleasure, or another Line be added; the Rectangle contained under the whole compounded Line and the Part added, together with the Square of half the given Line, is equal to the Square of the half Line and the Part added in one Line.



Let the given Line AB be bisected in C, and let BD be added to it, at pleasure.

I say, the Rectangle ADB added to CB Square is equal to the Square of CD.

On CD, describe the Square CDEF; and on AD the Rectangle AKLD; by drawing AK parallel to CE, and KL to AD, making DL equal to BD. Draw BG parallel to DE, and join FD.

Dem. The Rect. AI=IB (Ax.) and IB=GL (19. 1.) therefore, AI is equal to GL. - - - - - - - Ax. 3.
Conf. the Rect. AI+IB+BL=IB+BL+GL - Ax. 6.
And, if to both, be added the Square IFGH, of CB.
The Rect. AKLD+IFGH=CFED. - - - Ax. 2.
i. e. ADB▭, or AD×DB+CB□=CD□.

Or

Or it may be demonſtrated as the foregoing.

Produce BA, and make A*E* equal to BD.

Then is *E*D biſected, in C, and unequally cut, in B.

Wherefore, the Rect. *E*BD, under the unequal parts, (equal ADB) i. e. *E*B×BD,+CB □=CD □.

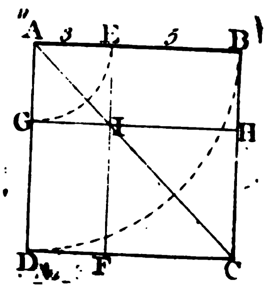
Let AB be 12, biſected in C, AC, 6, and CB, 6, and let BD, the part added, be 4.

Then, the Rect. AKLD,	i. e. AD×BD,	or	16× 4= 64
+ the Square IFGH,	i. e. CB □,	or	6× 6= 36
= the Square CFED,	i. e. CD □,	or	10×10=100

THEOREM VII.

If a Line be divided, equally or unequally, at pleaſure; the Square of the whole Line, added to the Square of either Segment, is equal to two Rectangles, under the whole Line and that Segment, together with the Square of the other Segment.

Let AB be divided, any how, in the point E. Then, the Square of A B, added to the Square of AE, is equal to two Rectangles under AB and that Segment, added to the Square of EB, the other Segment.



Deſcribe the Square ABCD; through E, draw EF parallel to AD; make AG equal AE, and draw GH, parallel to AB; and join AC.

Dem. The Rect. DI=IB (19. 1.) add GE to both; & DE=GB

But, the Rectangle DE+GB=DI+IB+2GE; - Th. 3.

and, if FH, i. e. the Square of EB, be added;

they are equal to the Square ADCB+GE. - - Ax. 2.

But, the Rectangles DE, GB, are under the whole Line, AB, and the Segment AE; for, AD=AB and AG=AE-Con.

Therefore 2BA×AE+EB □ =AB □ +AE.

Alſo, DH+FB+GE=DB+FH;

or 2ABE▭+AE □ =AB □ +EB □.

Let

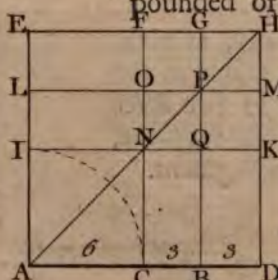
Let the whole Line, AB, be 8; let AE be 3, and EB 5.

Then the Square ABCD, of AB, i. e. $8\times8=64$ } $=73$
+ the Square AEIG, of AE, i. e. $3\times3=9$ }

But, the two Rects. DE, GB, i. e. 2 AB × AE, $8\times3=48$
+ the Square IFCH, of EB, ———— $5\times5=25$
is alfo $=73$

THEOREM VIII.

If a Right Line be divided, any how, in two Part[s], four Rectangles under the whole Line and eith[er] Segment, added to the Square of the other Seg[-]ment, is equal to the Square of a Line, com[-]pounded of the whole Line and the Segment firſt taken.



Let AB be divided, at pleafure, in C; and, if CB be the Segment taken, make BD equal to BC.

Then, four Rectangles under AB and CB added to the Square of the Segment AC, will be equal to the Square of AD.

On the whole Line, AD, conftruct the Square AEHD. Draw CF and BG, parallel to AE; make AI equal AC, and AL equal to AB, and draw IK, LM, parallel to AD.

DEM. Now CF & BG are parallel to AE; & IK, LM, to AD wherefore, FG, GH, HM, & MK, &c. are each equal CB. confequently, FP, GM, OQ, and PK are equal Square[s]. And, EO, LN, NB, and QD are equal Rectangles. But the Rect. EO+FP=ABC▭, or AB×BC. Conf. the four Rectangles EO, LN, &c. added to the fo[ur] Squares FP, GM, &c. are equal to four times AB×BC▭ and, if the Square IC (of AC) be added, they are = equal to the Square AEHD, of the Line, AD, com[-]pounded of AB and the Segment CB, equal BD. - Ax.

i. e. 4ABC ▭ or 4 times AB×BC+AC □=AD □.

Let AB be 9, divided at C, in 6 and 3.
and if BD be equal CB, then AD will be equal 12.

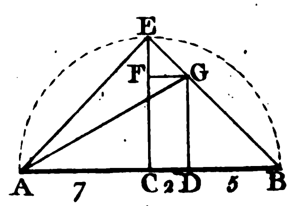
Then, the Rect. ABC, or $AB \times BC$, i. e. $9 \times 3 = 27$.
Conf. 4 Rectangles, $AB \times BC$ is equal to 108
$+$ the Square of AC, —— $6 \times 6 = 36$
$=$ the Square of AD, —— $12 \times 12 = 144$

THEOREM IX.

If a Right Line be divided, into two equal and two unequal Parts; the two Squares of the unequal Parts are, together, double the Square of half the Line, together with the Square of the intermediate Part.

Let AB be bisected in C, and cut unequally in D.

Then, the Square of AD, added to the Square of DB; is equal to twice the Square of AC, added to twice the Square of CD, the difference between AC and AD.



Draw CE perpendicular to AB, and equal to AC or CB.
Join AE and EB, and draw DG parellel to CE.
Through G, where DG cuts EB, draw FG parallel to AB; and, laftly, draw AG.

DEM. Now fince ACE is a right Angle, & CE is equal AC, the Angles CAE and AEC are half right. - C. 3. 10. 1.
And $AE\square = AC\square + CE\square$; i. e. equal $2AC\square$. - 20. 1.
For, ACE is a Right Angle, and $AC = CE$. - - Con.
And, becaufe CG is a Rectangle, $FG = CD$;
and FE is alfo equal CD; for EFG is a Right angle;
FEG (eq. FEA) is half right; th. the Angle $FEG = EGF$
confequently, FE is equal to FG (equal CD) C. 3. 9. 1.
and $EG\square = EF\square + FG\square, = 2\,CD\square$. - - - 20. 1.

 Now,

Now, AE□ = 2 AC□; and EG□ = 2 CD□ - above.
But AEG is a right Angle, and AG□ = AE□ + EG□;
i. e. equal 2 AC□ + 2 CD□.
Alſo, AG□ = AD□ + DG□; for ADG is a right Angle.
And DG = DB; for, GDB is a right Angle;
and DBG = DGB, equal CEB, half a Right one. - 4. 1.
Th. AG□, eq. AE□ + EG□, i. e. eq. 2 AC□ + 2 CD□;
is alſo equal to AD□ + DB, equal DG, □.
Therefore, AD□ + DB□ = 2 AC□ + 2 CD□. Q. E. D.

Let the whole Line AB be 14, divided equally, in C, and cut unequally, in D, in 9 & 5; AC 7, CD 2, & DB 5.

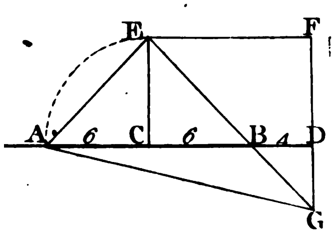
$$\left.\begin{array}{lll}\text{Then, AD}\square\text{, i. e.} & 9\times9=81 \\ +\ \text{DB}\square & 5\times5=25\end{array}\right\}=106$$

$$\left.\begin{array}{lll}\text{And AC}\square\text{, i. e.} & 7\times7=49 \\ +\ \text{CD}\square & 2\times2=\ 4\end{array}\right\}=53,\times2=106.$$

THEOREM X.

If a Right Line be biſected, and then produced at pleaſure; the Square of the whole compounded Line, together with the Square of the additional Part, will be double the Square of the half Line and the Part added, together with twice the Square of the half Line.

Let AB be biſected in C, and let BD be added.



I ſay, that the Square of AD added to the Square of BD, is equal to twice the Square of CD, added to twice AC Square.

Make ACE a Right angle, & CE eq. AC.
Compleat the Rectangle CEFD, and produce the Side FD.
Draw EB, cuting FD in G; and laſtly, draw AE & AG.

DEM.

DEM. Then, becaufe ACE is a R.Angle, (con.) & AC=CE
the Angles CAE and AEC are half Right. - C. 3. 10. 1.
And, for the fame reafon, CEB and EBC are half right.
And, becaufe CEFD is a Rectangle, EF=CD;
and fince EFG is a Right Angle, and FEG (eq. DBG, eq. EBC) is half right, FGE is half right. - - Ax. 3.
wherefore, FG=FE, eq. CD; and DG=DB - C. 3. 9. 1.
Now, fince ACE is a right Angle, and CE=AC
AE□=AC□+CE□; i. e. equal 2 AC Square. - 20. 1.
And, EG□,=EF□+FG□; i. e. equal 2 CD□.
But, AEG is a R. Angle; for AEC, CEG are half right.
wh. AG□,=AE□+EG□; i. e. equal 2 AC□+2CD□
But, ADG is a R. Angle; conf. AG□=AD□+DG;
and, fince BD=DG, BD□=DG□; - - - Ax.
Therefore, AD□+BD□=2 AC□+2CD□.

This Propofition may alfo be demonftrated as the former.

Let the given Line AB be bifected in C and let BD be added; alfo make A*E* equal BD in BA produced.

Then, becaufe *E*D, is bifected in C and cut unequally in B,
*E*B□+BD□=2CD□+2CB□.
But *E*B=AD; for A*E* was made equal to BD.
Wh: *E*B□+BD□ (eq. AD+BD□)=2CD□+2CB□.

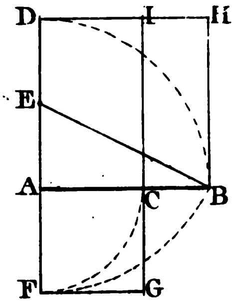
Let AB be 12; divided, in C, equally; and let BD be 4.

Then,	the Square of AD, i. e.	16×16=256	} =272
+	the Square of BD,	4× 4= 16	
But,	the Square of CD,	10×10=100	
+	the Square of AC,	6× 6= 36	
		136×2=272	

THEO-

THEOREM XI.

To divide a Right Line, in fuch wife, that the Rect-angle under the whole Line and one Segment, fhal be equal to the Square of the other; confequentl the greater Segment.



AB is the Line given, to be divided.

It is required fo to cut AB, in C, th the Rectangle, ABC, under the whole Li AB and the Segment BC, fhall be equal the Square of AC.

At the extreme A, draw AD perpen cular to AB.

Make AD equal AB, and bifect it in E; join EB. Produce DA. Make EF equal EB, and AC equal to A Then is AB fo divided, in C, that $AB \times BC = AC\square$.

Compleat the Squares ADHB and AFGC, and produ GC to I.

DEM. Then, becaufe AD is bifected in E, and AF is add the Rect. DFA, i. e. $DF \times AF + AE\square = EF\square$. - Tł But, $EF = EB$; wh. $EF\square = EB\square$; i. e. eq. $AB\square + A$' Conf. $DF \times AF + AE\square = AB\square + AE\square$. Take $AE\square$ from both, and there remains $DF \times AF = $/ But, the Rect. DIGF. is under the whole Line D' the Segment AF; and ADHB is the Square of A' wherefore, the Rect. $FI =$ the Square AH; and if tł ADIC, which is common to both, be taken awa is left the Rect. CIHB equal to the Square ACG i. e. the Rect. ABC, or $AB \times BC = AC\square$.

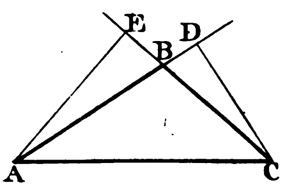
I

This Proposition cannot be exemplified in Numbers; i. e. the point C. cannot be found, nor the proof ascertained arithmetically; for there is no Number, whatever, can be so divided, that the product of the whole Line multiplied by one part, shall be equal to the Square of the other.

Therefore, a Line so divided is incommensurable.

THEOREM XII.

In an obtuse angled Triangle, the square of the Side which subtends the obtuse angle, exceeds the squares of the other two Sides, by two Rectangles, under either Side containing the obtuse Angle, and the part intercepted between the obtuse Angle, and a Perpendicular, let fall from the adjacent Angle, to the same Side, produced.



ABC is an obtuse angled Triangle.

Produce the Sides AB and CB; and to them, draw the Perpendiculars AE and CD.

I say, the Square of AC exceeds the Squares of AB and BC, by two Rectangles ABD, or CBE.

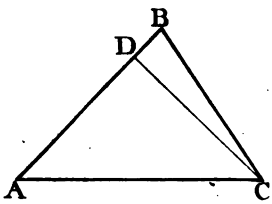
Dem. First, AC□ is equal to AD□+DC□. - - - 20. 1.
and AD□=AB□+BD□+2 ABD▭ - - - - - 4. 2.
wherefore, AC□=AB□+BD□+CD□+2 ABD▭.
But BC□ is equal to CD□+BD□.
Therefore, AC□=AB□+BC□+2ABD▭.

In the same manner, AC□ may be proved equal to AB□+BC□+2CBE Rectangles.

Consequently, the Rectangles ABD and CBE are equal.

THEOREM XIII.

In every Triangle, the Square of the Side ſubtending an acute Angle, is exceeded by the Squares of the other two Sides, which contains the acute Angle by two Rectangles under either of thoſe Sides, and a part of it, intercepted between the acute Angle and a Perpendicular, let fall from the oppoſite Angle.



In the Triangle ABC, draw a Perpendicular, CD, to the Side AB.

Then, the Square of AC, is exceeded by the two Squares of AB and BC, by two Rectangles, ABD, under the whole Side, AB, and the Segment BD.

And, $AB\square + AC\square = BC\square + 2\,BA \times AD$.

DEM. Firſt, $AB\square = AD\square + DB\square + 2ADB\sqsubset\!\sqsupset$. - Th. 4.

And, $BC\square = CD\square + DB\square$. - - - - - - - 20. I.

Wh. $AB\square + BC\square = AD\square + DC\square + 2DB\square + 2ADB\sqsubset\!\sqsupset$

But, $ADB\sqsubset\!\sqsupset + DB\square = ABD\sqsubset\!\sqsupset$. - - - - - Th. 3.

conſequently, $2\,ADB\sqsubset\!\sqsupset + 2\,DB\square = 2\,ABD\sqsubset\!\sqsupset$.

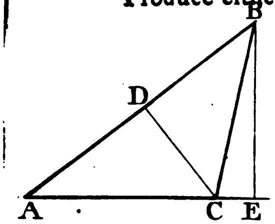
wherefore, ſubſtituting theſe for the other;

then, $AB\square + BC\square = AD\square + DC\square + 2\,ABD\sqsubset\!\sqsupset$.

But, $AC\square = AD\square + DC\square$. - - - - - - 20. I.

Therefore $AB\square + BC\square = AC\square + 2\,ABD\sqsubset\!\sqsupset$.

Secondly, thus, more briefly, when an Angle is obtuſe. Produce either Side, AC, and from B draw BE perpendicular to it.



Now, $AB\square = BC\square + AC\square + 2ACE\sqsubset\!\sqsupset$

Add, on both ſides, the Square of AC.

Then $AB\square + AC\square = BC + 2\,AC\square + 2\,ACE\sqsubset\!\sqsupset$. - - - - - - - Ax. 6.

But, $2\,ACE\sqsubset\!\sqsupset + 2AC\square = 2EAC\sqsubset\!\sqsupset$; or $2\,AE \times AC$. - - - - - - Th. 3.

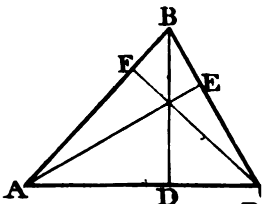
Therefore, $AB\square + AC\square = BC\square + 2\,AE \times AC$.

From which it is obvious, that the Rect. $BAD = EAC$.

By the first method it may be demonstrated; that the square of BC, is exceeded by the squares of AB and AC; by two Rectangles BAD; and by the second, the squares of AB and AC, exceeds the squares of BC, by two Rectangles EAC; therefore the Rectangle BAD is equal to EAC.

SCHOL. *From these two Theorems it is evident, that if the square of one Side of a Triangle exceeds the squares of the other two, the Angle subtended by that Side, and contained by the other two, is obtuse; and, if the square of one Side, be exceeded by the squares of the other two; it subtends an acute Angle; consequently, when the square of one Side is equal to the squares of the other two, the Angle it subtends is a Right one.*

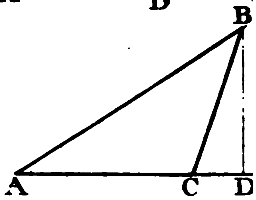
COR. Hence, the Perpendicular, and consequently the Area of any Triangle, may be found or obtained, by the measure of the Sides only.



In the Tri. ABC, AC□ +BC□ = AB□ +2 AC×CD. Wherefore, AC□ +BC□ −AB□ = 2 AC×CD; by Th. i. e. if AB□ be subtracted from the sum of AC□ +BC□, the remainder will be two Rectangles, under AC & DC. Wherefore, if half that Product be divided by the Side AC, the Quotient will be the Segment DC.

But BC□ = BD□ +DC□. Or, AB□ = AD□ +BD□.
wherefore, BC□ −DC□ = BD□. - - - - - 20. 1.
Conf. having subtracted the square of DC from BC □, or of AD from AB□, the remainder will be the square of the Perpendicular BD, the square Root of which, gives the Perpendicular required.

By the same means, either of the other Perpendiculars, AE or CF, may be found.



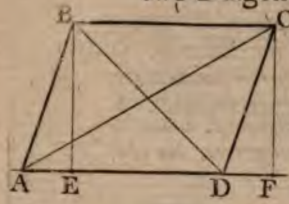
2ndly. When the Perpendicular falls without the Triangle, from the acute Angles of an obtuse angled Triangle, it is thus found.

In the obtuse-angled Triangle ABC, let fall the Perpendicular BD to the Side AC, produced.

Then, AB□ = AC□ +CB□ +2AC×CD. - - - - - Th. 12.
Wherefore AB□ −AC□ +CB□ = 2 AC×CD;
half of which Sum being divided by the Side AC gives CD.
But, CB□ = BD□ +CD (20. 1.) wh. CB□ −CD□ = BD□.
the square Root of which gives the Perpendicular BD.

THEOREM XIV.

In every Parallelogram, the ſum of the Squares of the two Diagonals is equal to the Squares of all the Sides, together.



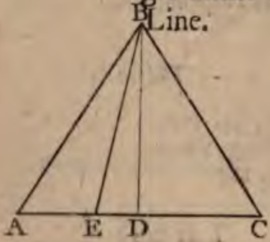
Let ABCD be a Parallelogram.

Draw the Diagonals, AC and BD; produce the Baſe AD, and draw the Perpendiculars BE and CF.

DEM. In the obtuſe angled Triangle ACD;
AC□=AD□+DC□+2ADF▭. - - Th. 12.
And, in the Tri. ABD; BD□=AB□+AD□−2DAE▭
But, EBCF is a Parallelogram, wh. EF=BC. - 15. 1.
and, AD=BC; wherefore AD=EF - - - Ax. 3.
conſ. AE=DF, ED being common; wh. ADF=DAE▭
Therefore, as much as AC□ exceeds AD□+DC□, viz. by the Rect. ADF, twice; by ſo much is BD□ exceeded by AB□+AD□, eq. BC□; viz. by 2DAE▭=2ADF.
Th. AC□+BD□=AB□+BC□+AD□+DC□.

THEOREM XV.

In Iſoſceles Triangles, the Square of one of the equal Sides is equal to the Square of any Line, drawn from the Vertex to the Baſe, added to a Rectangle under the Segments of the Baſe made by that Line.



In the Iſoſceles Triangle ABC, if BD be perpendicular, it biſects the Baſe, and is evident; for AB□=BD□+AD□; i. e. ADC▭, or AD×DC. - - 20. 1.

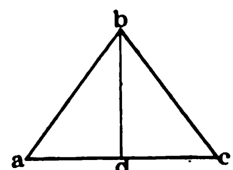
Let BE be drawn, at pleaſure.

I ſay, the Square of AB is equal to BE Square, added to the Rectangle AEC.

DEM. AB□ is equal to BD□ + AD□. - - - 20. 1.
and AD□ is equal to AEC▭ + DE□. - - - 5. 2.
(for AC, is cut equally in D, and unequally in E).
wherefore, AB□ = BD□ + DE□ + AEC▭.
But, BE□ = BD□ + DE□. - - - - - - - 20. 1.
Therefore, AB□ = BE□ + AE×EB, or AEC▭.

THEOREM XVI.

If a Perpendicular be drawn from any Angle of a Triangle to the oppoſite Side; the Squares of the Sides, containing that Angle, added to the Squares of the alternate Segments of the Baſe, are equal; and the difference between the Squares of the Sides, is equal to the difference of the Squares of the Segments.

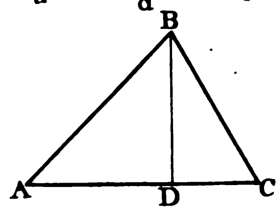


In Iſoſceles Triangles the thing is manifeſt.

In the Scalene Triangle ABC, draw the Perpendicular BD.

Then, the Square of AB added to DC Square is equal to BC Square added to AD Square.

And the difference, between the Squares of AB and BC, is equal to the difference, between the Squares of AD and DC.



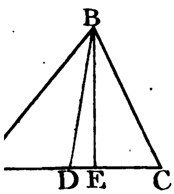
DEM. For, AB□ = AD□ + BD□.
And, BC□ = BD□ added to DC Square. - - 20. 1.
for the Angles, ADB and BDC, are Right ones. - Con.
Wh. AB□ + BD□ + DC□ = BC□ + BD□ + AD□.
that is, AB□ + BC□, is equal to BC□ + AB□. - Ax. 3.
Let there be taken, from both, BD□, which is common;
there is left AB□ + DC□ equal to BC□ + AD□.

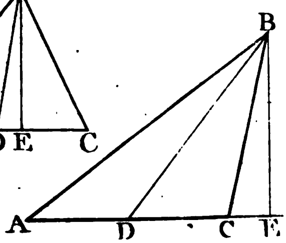
2nd. AD□ = AB□ − BD□; & DC□ = BC□ − BD□.
Therefore, AD□ − DC□ = AB□ − BC□.

THEOREM XVII.

If any Side of a Triangle is bisected, and a Right Line be drawn from the opposite Angle to the bisecting Point; the Squares of the other two Sides of the Triangle, will be equal to twice the Square of the bisecting Line, added to half the Square of the Side bisected.

If the Triangle be Isosceles, the thing is clear; for the Squares of the equal Sides, are equal to twice the Square of the Perpendicular, added to twice the Square of half the Base.





Let the Triangle ABC be Scalene; let the Side AC be bisected in D; and let BD be drawn.

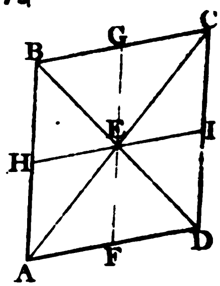
Then the Squares of AB and BC, together, are equal to twice BD Square, added to twice AD or DC Square.

Draw the Perpendicular BE.

DEM. AB□=AE□+BE□; and BC□=BE□+EC□.
Wherefore, AB□+BC□=AE□+EC□+2 BE□.
But AE□+EC□=2 AD□+2 DE□. - - 9. & 10.
Wh. AB□+BC□=2 AD□+2 DE□+2 BE□.
But, BD□=BE+DE□; conf.2BD□=2BE□+2DE□
Th. AB□+BC□=2 BD□+2AD□, equal ACD▭, or half AC□. Q. E. D.

By this Theorem, may also be demonstrated the 13th; viz. The Squares of the two Diagonals of every Parallelogram, are equal to the Squares of all the four Sides.

In the Par. ABCD (having drawn the Diagonals) through the Center, E, draw FG and HI, parallel to AB and AD.



In the Tri. AED, $AE\square + ED = 2EF\square + AF\square + FD\square$

conf. $AE\square + ED\square = AH\square + AF\square + FD\square + DI\square$.

And, $BE\square + EC\square = HB\square + BG\square + GC\square + CI\square$.

But, $AE\square + EC\square = \frac{1}{2}AC\square$; &, $BE + ED\square = \frac{1}{2}BD\square$.

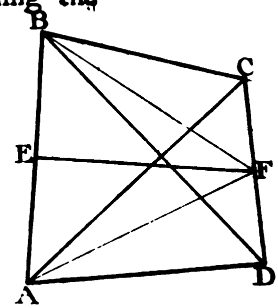
Also, $AF\square + FD\square = \frac{1}{2}AD\square$; & $AH + HB\square = \frac{1}{2}AB\square$.

and, the $\square$s of BG, GC, CI, & $ID = \frac{1}{2}BC + \frac{1}{2}CD\square$.

Th. $AC\square + BD\square = AD\square + AB\square + BC\square + CD\square$.

THEOREM XVIII.

In any Trapezium, if the middle Points of two opposite Sides be joined by a Right Line; the ſum of the Squares of the two other Sides, together with the Squares of the Diagonals, is equal to the ſum of the Squares of the biſected Sides, added to four times the Square of the Line joining the middle Points.



Let the Sides AB and CD of the Trapez. ABCD be biſected in E & F; and draw EF.

I ſay, the Squares of the two Sides, AB and CD, added to four times the Square of EF, is equal to the Squares of BC and AD, together with the Squares of AC and BD.

Draw AF and BF.

DEM. Now, $AF\square + BF\square = 2\,AE\square + 2\,EF\square$. - Th. 17.
and, $AD\square$ added to $AC\square = 2\,AF\square$ added to $2\,FD\square$;
alſo, $BC\square$ added to $BD\square = 2\,BF\square$ added to $2\,FD\square$.
conſ. $AD\square$ added to $AC\square$ added to $BC\square$ added to $BD\square$ is equal to $2\,AF\square$ added to $2\,BF\square$ added to $4\,FD\square$.
But, $AF\square$ added to $BF\square = 2\,AE\square$ added to $2\,EF\square$.
conſ. $2\,AF\square$ added to $2\,BF = 4\,AE\square$ added to $4\,EF$ Square.
But, 4 AE Square is equal to AB Square;
and, 4 FD Square is equal to CD Square. - Cor. to 4.

Therefore, $AD\square + AC\square + BC\square + BD\square = AB\square + CD\square + 4\,FF$ Square. Q. E. D.

ELEMENTS
OF
GEOMETRY.

BOOK III.

THE knowledge of the properties of a Circle is the ſubject of the third Book of Elements. As a Circle is the moſt perfect of Plane Figures, the properties peculiar to it, are moſt extraordinary: ſome of which are really ſurprizing, and are very extenſive in their application.

The general Definitions of a Circle, and its attributes, are ſo well known, and, as they are given in the General Introduction, in Def. 19, and the four following, it would be impertinent to repeat them here; ſave only what are particularly neceſſary. The moſt ſelf-evident Propoſitions I have reduced into Axioms, for the more expeditiouſly attaining the knowledge of other, more eſſential, Properties.

My chief aim, throughout the whole, being to render the Study of Geometry eaſy, and agreeable to young Students; for which end, I conſider, brevity, if clear, moſt conducive thereto; and am perſuaded, that if the knowledge of the moſt ſimple properties of Figures can be attained, by any other means, Demonſtration is uſeleſs.

If a perfect knowledge of all that is contained in the Axioms of this third Book, be not acquired, by a bare recital, and inſpection of the Figure, I ſhall pronounce that perſon's capacity inſufficient, to purſue the Study of the more ſublime properties of the Circle; and, if they are clearly evident without, of what uſe is Demonſtration? Where

Where the truth or evidence of an Aſſertion cannot be perceived, otherwiſe, Demonſtration is abſolutely neceſſary; but, where the evidence of ſight is ſufficient, the other is, in a great meaſure, uſeleſs.

There are ſome critical Geometricians who will not admit of any thing without Demonſtration; although they cannot but acknowledge many Propoſitions to be ſelf-evident truths. It would, perhaps, be no eaſy matter to give geometrical Demonſtration, that all Right Lines drawn from the Center of a Circle to the Circumference are equal; at the ſame time, no Perſon of common ſenſe can deny it, who knows any thing of the Figure, and geneſis of a Circle. If, therefore ſomething muſt be given or granted, why not others, which are as clear and ſelf-evident? for my part, I freely own, I can ſee no reaſon againſt it; and have, therefore, purſued the readieſt method for attaining the end aimed at; viz. to acquire a knowledge of Geometry in the moſt eaſy and familiar manner poſſible, by diveſting it of all that is ſuperfluous and unneceſſary; I mean unneceſſary Demonſtration, of what is clear without it.

I think it, however, neceſſary to apologize for the liberty I have taken in abridging it. Inſtead of 31 Theorems, in this third Book, according to Euclid, I have no more than 15, from Euclid; the reſt are diſpoſed of in the manner following.

The 2nd, the 5th and 6th, the 10th and 13th, are made Axioms.

The 9th is a Corollary to the 5th.

The 18th and 19th are Corollaries to the 8th; and the 23d and 24th may be deduced from the 3rd.

The 11th and 12th are both included in the 7th.

The 26 and 27th are the 2nd Corollary to the 9th; the 28th and 29th, Corollary 2nd of the 3rd; and the 37th is the 3rd of the 16th.

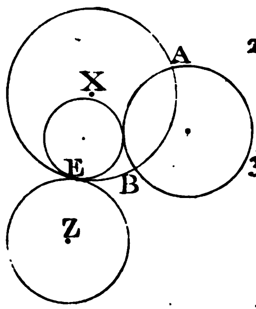
The ſix Problems are in Practical Geometry.

DEFI-

DEFINITIONS.

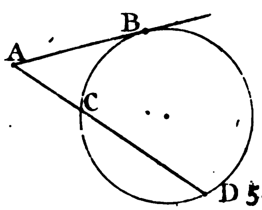
For the definition and geneſis of a Circle, ſee Def. 19. and 20, in the General Introduction.

DEF. 1. EQUAL CIRCLES are ſuch as have equal Diameters or equal Radii.



2. Circles are ſaid to cut one another, when their Circumferences croſs or interſect each other in two Points. A and B.

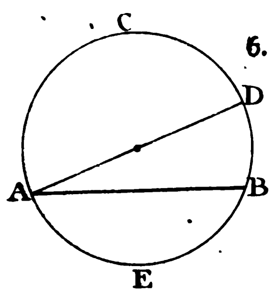
3. Circles are ſaid to touch, when their Circumferences meeting, either internally or externally, in any part, they do not cut each other. As X and Z touching in the Point E.



4. A TANGENT. A Right Line is called a Tangent, when, being in the ſame Plane with a Circle, it touches the Circumference, only, without cuting it. As AB in B.

And, the Point B, in which it touches the Circle is called the POINT OF CONTACT.

5. A SECANT is a Right Line, drawn from any Point without a Circle cuting the Circumference in two Points. As AD, in C and D.



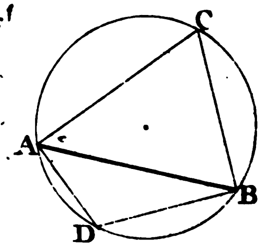
6. A SEGMENT, of a Circle, is any portion of a Circle, cut off by a Right Line; which is called a CHORD LINE, or SUBTENSE.
As AB making two Segments.

If the Segment be greater or leſs than a Semicircle (ACD or AED) it is called a GREATER or LESSER SEGMENT.

7. The Angle of a Segment, is the mixed Angle which is contained under the Chord Line and a portion of the Circumference. As ABC or ABE.

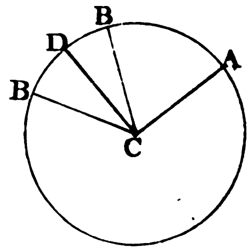
The

The Angle of a greater Segment is obtufe, of a leffer Segment it is acute.



8. An Angle in a Segment is that which is contained by two Right Lines, drawn from each extreme of its Bafe, or Subtenfe, to any Point in the ark of the Segment. As ACB or ADB.

9. Similar Segments are fuch as contain equal Angles, or whofe Angles are equal.



10. A SECTOR of a Circle, is comprehended between two Radii or Semi-diameters, and an Ark, or portion of the Circumference, intercepted between them. As AC, CB and the Ark AB.

If the Radii AC, CD contain a Right Angle, it is called a QUADRANT; as ADC.

11. An Angle in a Circle is faid to infift or ftand upon an Ark of the Circumference, which is oppofite to the Angle; or that portion of the Circumference below the Chord Line.

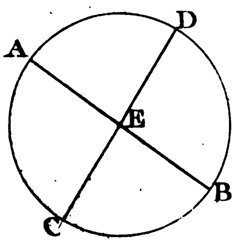
The Circumference, or curved Line which bounds a Circle, is concave towards the Center; and, externally it is convex.

AXIOMS.

The Six Axioms, or felf-evident Propofitions, which follow, contain properties of a Circle which are neceffary to be known, previous to the Demonftration of the Theorems which follow after. I call them Axioms, becaufe they are felf-evident; but fince that will not, by fome, be allowed, arbitrarily, I have endeavoured to illuftrate them, and to make the truth appear clear and manifeft.

AXIOM 1ft, All Lines drawn from the Center of a Circle to the Circumference are equal. As EA, EC, &c.

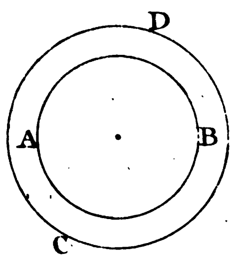
For, they are all Radii or Semidiameters.



2nd. Two or more Diameters, of a Circle, mutually bifect each other.

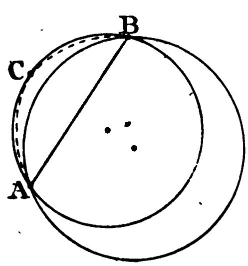
For they all pafs through the Center; confequently, they are bifected, in the Center. As AB and CD, in the Point E.

3rd. Circles, in the fame Plane, which cut, or touch each other, inwardly (outwardly it is mafeft) have not the fame Center.



For, if they have the fame Center, and equal Radii, they muft agree in every part. And, if they have the fame Center, and unequal Radii, they are parallel Circles, and can neither cut nor touch each other, in any part; for they are, every where, equi-diftant. As AB, and CD.

4th. Circumferences of Circles cut each other in two points only.



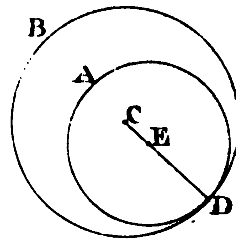
For, if it were poffible to cut, or touch only, in three Points, A, B, & C, each of the points of Section, will be equally diftant from the Center of each Circle; and confequently they will have the fame Center, which is contrary to the 3rd.

5th. A Right Line joining two Points (A and B) in the Circumference of a Circle, falls entirely within the Circle. This is manifeft.

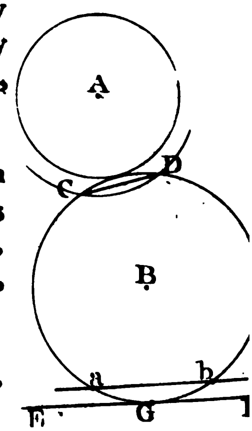
For if not, the Curve of the Circle, ACB, muft coincide in fome entire part, with the Line AB; which is contrary to the 19th Definition, in the General Introduction, and the Genefis of a Circle, in N. B. Def. 20.; from which it is manifeft, that the Curve of a Circle cannot fall in with a Right Line, in any part, it being uniform in every part.

6th. Cir-

6th. Circles touch each other, or a Right Line, in one Point only.



1. For, if the Circles, AD and BD, touched inwardly, in more than a Point, as at D, the Curve of the leſſer Circle, AD, muſt coincide in ſome part, entirely, with the curve of the larger Circle, BD, which, from the geneſis of a Circle, cannot be; ſeeing, they have unequal Radii, CD and ED; which produce different Curves, according to the Radius, which cannot, in any Part, fall into each other (for if they did they muſt coincide entirely in ſome part) therefore, they touch each other but in one Point.



2. If the Circles, A and B, touched outwardly, in more than a Point, they muſt cut each other; or their Circumferences will be a Right Line, in ſome part, as CD; which cannot be, for, it falls within both Circles; by the 5th.

3. The Circle, B, touches the Right Line, EF, but in one Point, at G.

For, if it touched in more than a Point, it muſt coincide, in ſome entire part, with the Right Line EF, which cannot be (Def. 19. and 20.)

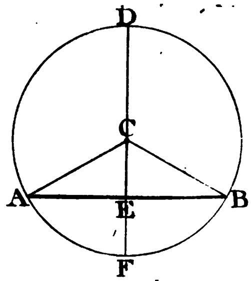
If any other Line, a b, be drawn between the Line EF and the Center, it will cut the Circle in two points of its Circumference, and the part, a b, of that Line falls within the Circle; by the laſt.

As this Property of Circles can only be known ſpeculatively, I preſume, that, what I have ſaid is as convictive, and ſatisfactory, as any Demonſtration whatever.

 THEO-

THEOREM I. 3. Euclid.

If a Right Line, drawn through the Center of a Circle, bisects a Chord Line, not drawn through the Center, it will cut it perpendicularly.



In the Circle ADB; let the Right Line DE pass through C, the Center, dividing the Chord Line AB into two equal parts, at E.

I say, the Line DE will be perpendicular to AB.

From the Center, C, draw CA, CB.

DEM. The Triangles ACE, ECB are equilateral and equiangular to each other.

For, CE is common to both; AE=EB, - - Hyp.
and AC is equal to CB. - - - - - Ax. 1.
Wh. because AC=CB the Angle CAE=CBE.- Th.9.1.
Also, the Angle AEC=CEB. - - - - - - 8. 1.
consequently, they are Right ones. - - - C. 2. 1. 1.
Therefore, DCE is perpendicular to AB-Def. 10. & 11. 1.

COR. 1. A Chord (AB) being bisected by another Right Line, at Right Angles with it (in E) the perpendicular Line, (DE) will pass through the Center of the Circle.

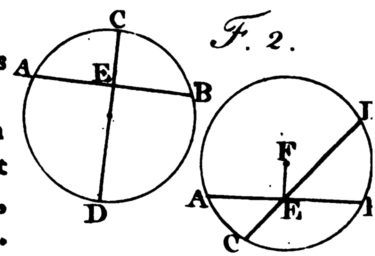
COR. 2. A Perpendicular drawn from the Center of a Circle to a Chord Line, bisects the Chord; and also the Ark of that Chord. As AFB, in F.

Hence, a Right Line, or an Ark of a Circle, is bisected. Prob. 8th.

THEO-

THEOREM II. 4. Euclid.

If, in a Circle, two Chord Lines cut each other, and are not both drawn through the Center, they cannot bisect each other.



F. 2.

Let AB and CD be two Chord Lines cuting each other, in E.

If one of them, CD (Fig. 1.) passeth through the Center, it is evident, that it cannot be bisected by the other, AB, which does not pass through the Center.

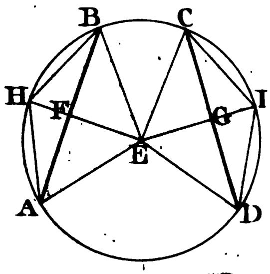
If neither of them passes through the Center, one of them may be bisected, but both cannot.

DEM. For, if AB (Fig. 2.) is bisected, in E, a Right Line, FE, joining the Center, F, and the point of Section, E, will be perpendicular to AB. - - - - - Th. 1.
consequently, AEF & FEB are Right Angles. - Def. 10. 1.
And, if CD be also bisected, in E;
FE is, also, perpendicular to CD. - - - - Th 1.
Wherefore, CEF and FED are Right Angles. - - Sup.
But, AEF, FEB are Right Angles. - - - proved.
consequently, CEF, FED are not Right Angles.- Ax. 2. 1.
for it is evident, that one is greater and the other less.
Therefore, CD is not bisected in E. Q. E. D.

THEO-

THEOREM III. 14. Euclid.

Equal Chord Lines, in a Circle, are equally diſtant from the Center; and Chord Lines which are equi-diſtant, are equal.



Let AB and CD be equal Chord Lines in the Circle ADB.

I ſay, they are equally diſtant from the Center, E.

From E, the Center, draw EF and EG perpendicular to AB and CD; and join AE, EB, EC, and ED.

DEM. Now, AB is equal to CD, by the Hypotheſis.
and AE, EB, EC, and ED are all equal. - - Ax. 1.
wh. the Triangles AEB, CED are congruous. - Def. 44.
(for, they are equilateral, to each other, by Conſtruction)
conſequently, they are equiangular, - - - 7. 1.
And, being alſo Iſoſceles, the Angles at A and B, C and D, are all equal amongſt themſelves. - - - 9. 1.

Now, ſince AB & CD are biſected in F & G, AF=DG.
Wherefore, in the Triangles AFE, EGD, the Sides, AE, AF, are equal to ED, DG, reſpectively;
and they contain equal Angles, FAE=EDG. - proved.
Therefore, EF is equal to EG. - - - - - 8. 1.

2nd. AB and CD are two Chord Lines, equally diſtant from the Center of the Circle ADB. I ſay, AB=CD.

Draw the equal Lines, EF, EG, perpendicular to AB and CD; which will biſect them, in F and G.- Cor. 2. 1.
Draw AE and ED.

DEM. Becauſe AE=ED, AE□=ED□; &, EF□=EG□.
But, AE□=AF□+EF□; & ED□=EG□+GD□.
conſ. AF□+EF□=EG□+GD□; wh. AF□=GD□
Therefore, AF=GD (Ax. B. 2.) & conſ. AB=CD - Ax. 5. 1.

The ſecond Part of this is the converſe of the former; which, I ſhould have made a Corollary to it; but on account of the different manner of Demonſtration; by the laſt, of which, Euclid demonſtrates both.

COR. 1. Fqual Chord Lines, in the ſame or equal Circles, ſubtend equal Angles at the Center.

For, ſince AB=CD, and AE, EB, EC, and ED are all equal, the Triangles AEB, CED are congruous; conſ. the Angle AEB=CED. - - - - 7. 1.

And, ſince equal Circles have equal Radii, the Angles which equal Chords ſubtend, in equal Circles, are alſo equal.

COR. 2. Equal Chords, in the ſame or equal Circles, ſubtend equal Arks, and cut off equal Segments. And, equal Arks have equal Chords.

AB=CD; and, ſince EF=EG, if EF and EG be produced, to H and I, FH=GI. - - - - Ax. 7. 1.
And, ſince EF and EG are perpendicular to AB and CD, the Arks AHB, CID, are biſected, in A and I. - C. 2. 1.
wherefore, the Triangle AHB=CID;
the Arks, AH, HB, CI, and ID, are all equal, and the whole Ark AHB is equal to CID. - - Ax. 5. 1.

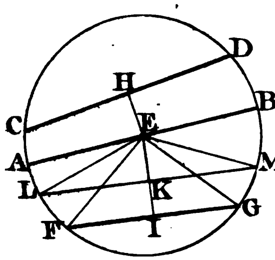
For, if AB was applied to CD, being equal, they would perfectly agree; the Point A with D, and B with C; alſo, the Point H would agree with the Point I, and every other Point, in the Ark AHB, with a correſponding Point, in the Ark CID; conſequently, the whole Ark, AHB, would coincide with the Ark CID; therefore the Segment AHB=C.D; and, conſ. DAHC=ADIB.

The ſame may be ſaid of equal C.rcles, having equal Radii.

THEO-

THEOREM IV. 15. Euc

A Diameter is the greateſt Right Line, which can drawn in a Circle; and, of all other Chord Lir that is the greateſt which is neareſt to the Cente



Let AB be a Diameter, and E the C ter of the Circle CDG.

CD and FG are Chord Lines, at differ diſtances from the Center.

I ſay, that AB is the greateſt; and F the fartheſt off, is the leaſt of the three

Draw the Perpendiculars EH and to CD and FG, from the Center.

DEM. Becauſe FG is farther from the Center than C EI is greater than EH. (3.) Make EK equal te EH. Draw KL, perpendicular to EK, and produce LK to and, laſtly, draw EL, EM, EF, and EG.

Now, becauſe EH is equal to EK, and are perpen cular to CD and LM; LM is equal to CD. - Th. But AE+EB (eq. LE+EM) is greater than LM. - 13. Therefore, AB is greater than LM; i. e. than CD, eq. LM

2nd. In the Triangles LEM, FEG, the two Sides LE, EM are equal to the two Sides FE, EG, and the Angle LEM is greater than FEG.

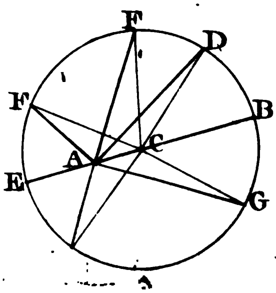
Therefore, FG is leſs than LM; i. e. than CD, whi is equal to LM. - - - - - Cor. to 8.

THE(

THEOREM V. 7. Euclid.

If any Point, except the Center, be taken within a Circle, and, from that point, divers Right Lines be drawn to the Circumference; the greateſt of thoſe Lines is that which paſſes through the Center; and, the leaſt is the remainder of that Line, produced to the Circumference.

Of all others, drawn from that Point, that which falls neareſt to the Line, paſſing through the Center, is greater than the more remote ones; and, but two equal Right Lines can be drawn, from that Point, to the Circumference.



Let A be the Point aſſumed, in the Circle EFG; from which draw AB, AD, AE, &c.

Firſt; AB, which paſſes through the Center; C, is greater than AD, or AF, or any other Line, which can be drawn from the point A. Draw DC.

DEM. In the Triangle ADC, the two Sides, AC, CD, are greater than the remaining Side, AD. - - 13. 1.
But, CD is equal to CB, - - - - Ax. 1.
wherefore, $AC+CD=AC+CB$, i. e. AB. - Ax. 6. 1.
Therefore, AB is greater than AD.

2nd. AE is the ſhorteſt Line, which can be drawn from the Point A. Draw any Line, AF, and join FC.

Then, in the Tri. AFC, $AF+AC$ is greater than CF.
But, CF is equal to CE, - - - - Ax. 1.
Wh. (taking away AC) AF is greater than AE - Ax. 8. 1.

3rd. That Line is the longeſt, which falls neareſt to AB.

In the Triangles ADC, AFC, AC, CD are equal to AC, CF, reſpectively; for AC is common, & CD=CF - Ax. 1. But, the Angle ACD is greater than ACF. - Ax. 2. 1. Therefore, the Side AD is greater than AF. - Cor. to 8. 1.

4th. No more than two equal Lines can be drawn, from the Point A, to the Circumference.

For, ſince AF is proved leſs than AD, and AB greater, every Line which can be drawn, from A, between D and F, will be leſs than AD, and between B and D greater; wherefore, no other equal Line can be drawn on that Side EB.

Make the Angle BCG=BCD; (Pr. 4.) and draw AG. Then, becauſe the Angle BCG=BCD, GCA=ACD. For, BCG+GCA=BCD+DCA. - Th. 1.1. & Ax. 9. And, becauſe CG=CD, and AC is common; we have AC, CG, reſpectively equal to AC, CD; and the Angle ACG=ACD; therefore AG=AD - 8. 1.

And, as no other Line, equal to AG, can be drawn on that Side BE; conſequently, no more than two equal Lines can be drawn from the ſame Point, A, to the Circumference.

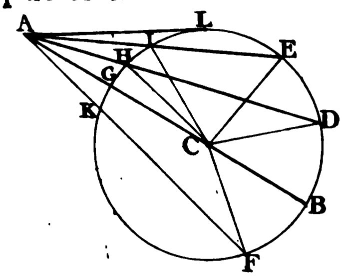
Cor. Hence it is evident, that, if from any Point in a Circle, more than two equal Lines can be drawn to the Circumference, that Point is the Center.

THEO-

THEOREM VI. 8. Euclid.

If from any Point without a Circle, Right Lines are drawn to the concave part of the Circumference; that which passes through the Center, is the greatest; and that Line which falls nearest to it, is greater than that which is more remote.

Of those, which fall upon the convex part of the Circumference; that is the least, which, if produced would pass through the Center; and that Line, which falls nearest to it, is less than any other, further off. And, no more than two equal Lines, can be drawn from any Point, without the Circle, either to the convex or concave part of the Circumference.



Assume the Point, A, at pleasure, and draw AB, AD and AE.

First; AB which passes through the Center, C, is greater than AD or AE, not passing through the Center. Draw CD.

Dem. Then, in the Triangle ACD, the two Sides AC, CD are greater than AD. - - - - - - Th. 13. 1. But, CD=CB; wherefore, AC+CB, i. e. AB, =AC+CD; consequently, AB is greater than AD.

2nd. AD is greater than AE. Draw CE.

Then, in the Triangles AEC, ADC, the Sides AC, CE, are respectively equal to AC, CD. - - - Ax. 1. And the Angle ACD is greater than ACE. - - Ax. 2. 1. Therefore, AD is greater than AE. - - Cor. to 8. 1.

3rd. AG is leſs than AH or AI. Draw CH and CI.

Then, in the Tri. ACH; AH+HC is greater than AC.
But, GC is equal to HC; - - - - Ax. 1.
wherefore, AC—GC, i. e. AG, is leſs than AH - Ax. 8. 1.

4th. AH is leſs than AI, which falls further from AC.

For, in the Tri. AIC, the two Sides, AI, IC, are greater than AH, HC, to any Point, H, within the Triangle - 14. 1.
But IC=HC; wherefore, AH is leſs than AI. - Ax. 8. 1.

5th. No more than two equal Lines can be drawn from any point (as A) without the Circle, either to the convex or concave part of the Circumference.

For, ſince all Lines drawn from A, between G and H, are leſs than AH, and between H and I, greater; no other Line, drawn on that Side AB, can be equal to AH.

But an equal Line, AK, may be drawn on the other Side.

So likewiſe; all Lines, drawn from A to the concave Periphery, between AB and AD, are greater than AD; and between AD and AE, leſs; conſequently, no other Line can be drawn, on that Side AB, equal to AD.

But, if BF be made equal BD, and AF be drawn, AF is equal to AD. Draw CF; which is equal to CD.

For, the Angle FCB=BCD, by Conſtruction, wherefore, FCA=ACD - - Th. 1, 1. & Ax. 7. 1.
Wh. in the Triangles ADC, ACF, the Sides AC, CD are reſpectively equal to AC, CF; for CF, CD are Radii; and, the Angle ACD=ACF, therefore AF=AD - 8. 1.

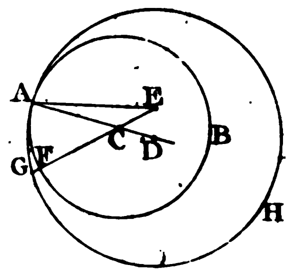
No other Line, equal to AF (eq. AD) can be drawn on that Side of AB; wherefore, but two equal Lines can be drawn from any Point, A, without the Circle, either to the convex or concave part of the Circumference.

Cor. The greateſt Right Line that can be drawn, from a Point without a Circle, to the convexity of the Circumference, is equal to the leaſt drawn to the concave part.

For they unite in a Tangent to the Circle, at L.

THEOREM VII. 11. 12. Euclid.

If two Circles touch each other, a Right Line, joining their Centers, will pass through the Point of contact of the two Circles.



First; let AB and AH be two Circles, touching each other, inwardly, in A. C is the Center of the lesser Circle, AB.

Draw the Right Line AC; it will, if produced, pass through D, the Center of the other Circle.

If not; let any other Point, as E, be the Center of AH, and draw EC, cuting the two Circles in F & G, and join AE.

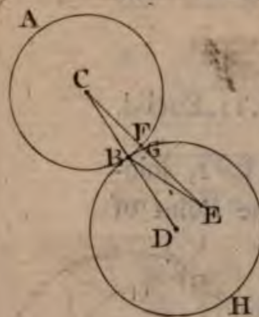
DEM. Then, because C is the Center of the Circle AB, CA is equal to CF. - - - - - - Ax. 1.
If CE be added to both, then CE+CF=CE+CA - 6. 1.
But, CE+CA is greater than AE. - - - Th. 13. 1.
and, EA is equal EG (by the Supposition). Ax. 1.
wherefore, EG is less than EF (the greater than the less) which is absurd, and cannot be.
Therefore, E is not the Center of the Circle AH; consequently D is the Center; and, the Right Line DC passes through the Point of contact, A.

2ndly. AB and BH are two Circles, touching each other, outwardly, at B; C is the Center of AB.

Draw the Right Line CB; which, if produced, will pass through the Center, D, of the other Circle.

If it be denied, let E be supposed to be the Center of BH, and draw CE and BE.

 Then

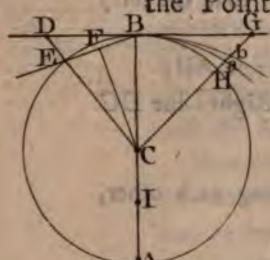


Then, becaufe C is the Center of the Circle AB, CF is equal to CB;
and, if E be the Center of BH, EG=EB.
Wherefore, CF+GE=CB+BE. - Ax. 6. 1.
And, if FG be added to the former; CE, i. e. CF+FG+GE, is greater than CB+BE.
But, CB+BE is greater than CE - Th. 13. 1
wherefore, BC+BE is both greater and lefs than CE, which is abfurd, and cannot be.
Therefore, E is not the Center of the Circle BH;
confequently D is the Center: and, the Right Line, CD, joining the two Centers, C and D, paffes through B, the Point of Contact.

COR. Circles touch each other, either inwardly or outwardly, in a Point only.

THEOREM VIII. 16. Euclid.

If a Right Line be drawn, through the extreme point of a Diameter of a Circle, at right angles with the Diameter, it will fall wholly without the Circle; and no other Right Line can be drawn, from the Point of contact, between the Tangent and the Circle.



Let AB be a Diameter, and DG a Right Line drawn at right angles with AB, through the extreme B.
Draw CD cuting D in DG.
I fay, the Point D is without the Circle.

DEM. In the Tri. CDB, the Angle DBC is a right one;
conf. BDC is acute; wh. DC is greater than BC. - 12. 1.
But, the Point B, is in the Circumference.
wherefore, the point D is beyond the Circumferenee - Ax. 1.
confequently BD falls wholly without the Circle.

Or thus after Euclid.

If BD does not fall without, it will cut the Circle; ſuppoſe in B and E. Draw EC.

Then, in the Triangle EBC, EC=CB; - Ax. 1.
wherefore, the Angle CEB=CBE.
But, CBE is a right angle (Hyp.) conſ. CEB is a right one.
And the two angles, CEB, EBC, of the Triangle CEB, are equal to two Right Angles.
But all the three Angles of every Triangle=two R. angles.
Therefore, BE, i. e. BD, does not cut the Circle, but muſt neceſſarily fall without it.

2nd. Let BE be drawn (if poſſible) between DB and the Circumference of the Circle.

Now, ſince CBD is a Right Angle, CBE is acute.
Let CF be drawn perpendicular to BE.
Then, CF, ſubtending an acute Angle, is leſs than CB.
But, the Point B is in the Circumference.
conſ. CF does not reach the Circumference. - Ax. 1.
wherefore, the Point F is within the Circle.
Therefore, the Right Line BE cannot be drawn between the Tangent, BD, and the Circumference.

COR. 1. A Tangent touches a Circle in one Point only. For if it touched in two Points it would cut the Circle.

COR. 2. A Right Line perpendicular to the Tangent, at the Point of Contact, paſſes through the Center.

COR. 3. A Right Line drawn from the center of the Circle to the Point of contact, of a Right Line touching the Circle, is perpendicular to that Tangent.

COR. 4. The Angle of a Semicircle is a right one.
For, it is greater than any right-lined acute angle.

N. B.

N. B. The external Angle, made by a Tangent and the adjoining Ark of the Circle, is leſs than any right-lined Angle whatever.

For it has been proved, that a Right Line cannot be drawn, from the Point of contact, between the Tangent and the Circumference; wherefore, the Angle GBH cannot be made leſs by a Right Line, ſeeing it will cut the Circle, and conſeqnently make a larger Angle, at ſome other Point in the Circumference.

Yet may the Angle GBH be leſſened infinitely, by curved Lines; which is, apparently, a Paradox.

For ſince the Angle of a Semicircle is Right, and the Tangent BG is perpendicular to the Diameter, AB, there cannot be left any remainder of the Angle as a Compliment to it; becauſe, the Compliment of an Angle is its deficiency of a Right Angle; or (being obtuſe) to two Right Angles.

Now, ſince it has been proved that a Right Line cannot touch a Circle but in one Point only; it muſt hold equally true, in reſpect of a large Circle as of a ſmall one.

Wherefore, if any other Radius, greater than BC, be taken, as BI, and an Ark, Ba, be drawn, it is evident that the Angle GBa is leſs than GBH.

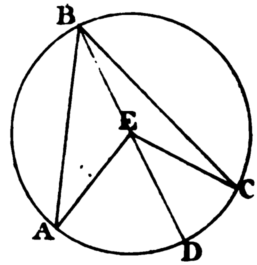
If a larger Radius, BA, be taken, and, on the Center A, an Ark, Bb, be drawn, the Angle GBb is leſs than GBa; notwithſtanding, the Angles, ABH, ABa, and ABb, are the ſame (by the 4th Cor.) and it is evident, that if a ſtill larger Radius be taken, the Angle GBb may ſtill be leſſened infinitely.

SCHOL. *Hence it is manifeſt, that any Right Line,* GH, *is infinitely diviſible, by enlarging the Radius* BA *infinitely; ſeeing that, the Circumference of a Circle cannot coincide with two Points,* B *and* G, *of the Right Line* BG.

THEO

THEOREM IX. 20. Euclid.

An Angle, at the Center of a Circle, is double of an Angle touching the Circumference, when both ſtand on the ſame Ark.

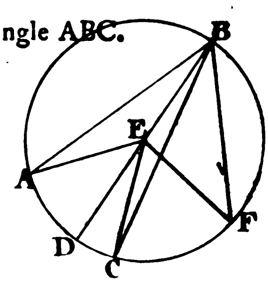


In the Circle ABC, let AEC be an Angle at its Center, and ABC is an Angle touching the Circumference; both ſtanding on the ſame Ark AC.

I ſay, the Angle AEC is double the Angle ABC.

Draw the Right Line BE, and produce it to D.

DEM. In the Triangle ABE, becauſe the ſide AE = EB,
the Angle A is equal to ABE - - - - - - - 9.1.
But, the external Angle AED = the Angle A + ABE - 10.1
Therefore, the Angle ABD = half the Angle AED
Alſo DBC = ½DEC; for the Triangle EBC is Iſoſceles.
Conſ. AED + DEC (i. e. AEC) = twice ABD + DBC;
i. e. equal to twice ABC.
Therefore, the Angle AEC is double the Angle ABC.



Caſe 2nd. When the Angle CEF, at the Center, falls without the Angle CBF, at the Circumference. Draw BED, as before.

Now, the external Angle, DEC, = ECB + EBC; (10. 1.) and ECB = EBC - 9.1
wherefore, DEC is equal to twice DBC.
And, the Angle DEF = EFB + EBF;
therefore. equal twice DBF.
Conſequently, DEF − DEC, i.e. CEF, = 2DBF − 2DEC, i.e. 2CBF. Therefore, the Angle CEF is double CBF.

 Cor.

Cor. 1. The Angle at the Circumference, ſtanding on an Ark, is equal to an Angle at the Center, on half that Ark.

For, if the Ark AC be=CF, the Angle AEC=CEF. But, the Angle AEC is double ABC, &, CEF double CBF consequently the Angle AEC (eq. CEF)=ABC+CBF i. e. the Angle ABF, on the Ark ACF, is equal to AEC (or CEF) on AC, or CF, half that Ark.

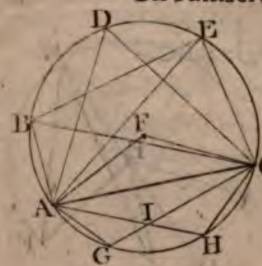
Cor. 2. In the ſame or equal Circles, Angles ſtanding on equal Arks, whether they be at the Center or at the Circumference, are equal to one another.

This is evident from the laſt; for the Ark AC being equal to CF, the Angle AEC=CEF; and, the Angle ABC (eq. half AEC)=CBF, half CEF.

And, becauſe the Radii of equal Circles are equal, the Angles on equal Arks, in equal Circles, are alſo equal. And converſely, equal Angles ſtand on equal Arks.

THEOREM X. 21.Eucli

All Angles, which ſtand on the ſame Ark, or are in the ſame Segment of a Circle, and touching the Circumference, are equal to one another.



In the Segment ADC, the Angles ABC, ADC, and AEC, are all equal.

To the Center, F, draw AF and CF making the Angle AFC.

Dem: Then, the Angle AFC, at the Center, is double AEC, at the Circumference. For, they ſtand on the ſame Ark, AGC, or Chord, AC. But the Angle AFC is double ABC, or any other Angle ADC, touching the Circumference. - - - - Th. * Wherefore, the Angles ABC, ADC, &c. being each equal half AFC, are, therefore, equal amongſt themſelves.

* Theory of Plane Angles, Art. 4. Page 23.

If the Segment be a Semicircle, or leſs than a Semicircle, it may be demonſtrated after this manner.

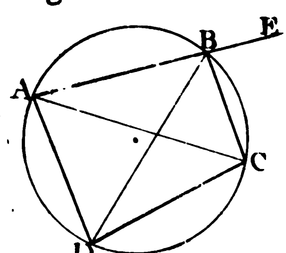
AGC and AHC are Angles in a leſſer Segment, AGHC. The Angle AIC (being external, in reſpect of the Triangles GAI & ICH) = the Angle G + GAI; and alſo, to H + HCI; wherefore, the Angle G + GAI = H + HCI. But, the Angle GAI, i. e. GAH = HCI, i.e. HCG; above. Therefore, the Angle AGC = AHC. Q. E. D.

COR. Right Lines, BC and AH, joining the extreme Points of two equal Arks, AB and CH, in the Circumference of a Circle, ſo as not to croſs each, are parallel.

For, having drawn AC) the Arks AB, CH being equal, the Angle BCA is equal to the Angle CAH. - C. 2. 9
Therefore, BC is parallel to AH. - - - - Th. 4. 1.

THEOREM XI. 22. Euclid.

In every Quadrilateral, inſcribed in a Circle, the oppoſite Angles are equal to two Right Angles.



ABCD is a Trapezium incribed in a Circle.

I ſay, the Angles A and C, alſo B and D, are, together, equal to two Right ones.

Draw the Diagonals AC and BD,

DEM. Then, in the Triang. ABD, the Angle DAB + ABD added to BDA are equal to two Right ones. - - 10. 1.
But, the Angle BCA = BDA; and ACD = ABD. - 10.
Wh. BCA + ACD, i. e. BCD, + BAD = two Right Angles

COR. 1. The external Angle, CBF, made by producing any Side, AB, of a Quadrilateral inſcribed in a Circle, is equal to the oppoſite Angle, ADC: - by Theo. & 1. 1.

2. Every Parallelogram inſcribed in a Circle is a Rectangle.

THEOREM XII. 31. Euclid.

An Angle, at the Circumference, in a Semicircle, is a Right Angle.

In the Circle ABCD, let AC be a Diameter, and E the Center.

To any Point, B, in the Circumference, draw AB and CB.

I say the Angle ABC is a Right Angle.

Draw EB; the Triangles, AEB, BEC, are Isosceles. AE, EB, and EC being equal.

DEM. Then, because AE=EB, the Angle EAB=ABE and, because EB=EC, the Angle EBC=BCE - 9. 1.
wherefore, the Angle ABC=BAC+ACB - - Ax. 6.
But, if one Angle of a Triangle be equal to the other two, it is a Right Angle. - - - - C. 1. 10. 1.
Therefore, ABC is a Right Angle Q. E. D.

Or, if AB be produced, the external Angle CBF is equal to BAC + BCA - - - Th. 10. 1.

It may be otherwise proved, after this manner.

Draw ED perpendicular to AC, and join BD.

Now, the Angle AED is double the Angle ABD.
And the Angle DEC is double the Angle DBC. - Th. 9.
But, the Angles AED, DEC, are Right ones. - Con.
wherefore ABD, DBC are each half Right - C. 1. 10. 1.
Conf. ABC, i. e. ABD+DBC, is a Right Angle.

Or, the Angle ABC at the Circumference, standing on the Ark or Semi-circumference ADC, is equal to an Angle at the Center, AED or DEC, (which are Right Angles) on half that Ark. - - - - - - Th. 9.

C R

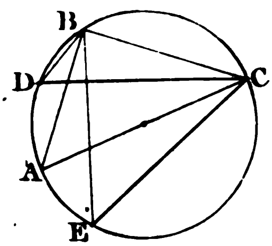
From this Theorem is deduced that moſt elegant and expeditious Method for making a Right Angle, or drawing a perpendicular at the extremity of a Right Line; Prob. 10. Method. 1ſt.

COR. An Angle in a Segment, leſs then a Semicircle, is obtuſe; and an Angle in a greater Segment is acute.

In the Segment ADCB ſeeing that the Angle ABC is right, ACB, in that Segment, is acute. - - - - C3. 10. 1.
And, the Angle AGB+ACB=two Right Angles - Th. 11.
But ACB is acute; wherefore AGB is obtuſe. - Def. 13.

THEO. The Angle in a leſſer Segment exceeds, and the Angle in a greater, is deficient of a Right Angle, by the Angle made between the Chord of the Segment and a Diameter, drawn from either extreme of the Chord.

Let AC be a Diameter, and ABC is a Right Angle. Draw two Chords, DC and EC, and join DB and EB.



Then, in the leſſer Segment DBC, the Angle DBC is greater than the Right Angle ABC, in a Semicircle. - Ax. 1. And the difference is the Angle D B A,

But, the Angle DBA=DCA.-Th. 10. Therefore, the Angle DBC, in a leſs Segment, exceeds a Right Angle, by the Angle ACD.

2nd. In the greater Segment EDBC, the Angle EBC is leſs than a Right Angle ABC. And, the deficiency is the Angle ABE, equal ACE. Therefore, the Angle, EBC, in a greater Segment, &c.

THEO-

THEOREM XIII. 32. Eucl

The Angle made between a Tangent and a Rig Line, cuting the Circle, drawn from the Point Contact, is equal to the Angle made in the o posite Segment.

Let ABC be a Right Line, touching t Circle BDE, in B.

Draw at pleasure, from B, the Chord B

The Angle ABD is equal to DEB, any Angle made in the Segment DBE.

And the Angle DBC is equal to DG in the other Segment, DBG.

Draw BF perpendicular to AC, and join FE.

DEM. Now BF is a Diameter of the Circle. - - C. 2
wherefore, FEB is a Right Angle. - - - - Th.
But, ABF is a R. Angle (Con.) conf. FEB=ABF, Ax.
But, the Angle FED=FBD (10) Th. DEB=DBA, Ax.

2nd. Let BG be the Chord of the Segment GFE
Draw GE, making an Angle, GEB.

Then, the Angle GDB (eq. GEB)=GBA; as abov
And, the Angle G+GDB+DBG=two R. Angles. -10.
But, the Angle ABD+DBC= two Right Angles. - 1.
and the Angle GDB (eq. GBA)+DBG=DBA. - Ax. 2.
Therefore the remaining Angle, DGB=DBC. - Ax. 7.

Or, in the Quadrilateral DEBG, the oppofite Angl DGB, DEB, are equal to two Right Angles. - Th. 1
And the Angle ABD+DBC=two Right Angles. - 1.
But, the Angle DEB=DBA, (as above) Th. DGB=DB

THE

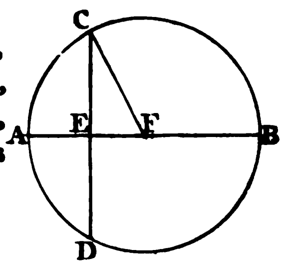
THEOREM XIV. 35. Euclid.

In Circles, if two Chord Lines cut each other; the Rectangle, contained under the Segments of one Line, will be equal to the Rectangle under the Segments of the other.

1st. When both pass through the Center, it is evident.

For, the Rectangle under the Segments of each, is the Square of the Radius; consequently they are equal.

2nd. In the Circle ACBD; if AB, passing through the Center, bisects CD, which does not pass through the Center, in E; then, the Rectangle AEB, is equal to the Square of CE, or ED.

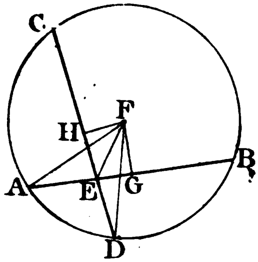


Let F be the Center. Draw CF.

DEM. Then, because AB is bisected, in F, and cut unequally, in E; AE × EB + EF□ = FB□ - - 5. 2.
i. e. = CF□, = CE□ + EF□. - - - - - 20. 1.
for, CEF, is a R. Angle (C. 2. 1.) and CF = FB. - Ax. 1.
Wherefore, AE × EB + EF□ = CE□ + EF□.
Take away from both, EF□, which is common;
There remains AE × EB = CE□. i. e. ▭AEB = ▭CED.

3rd. When

3rd. When, neither, of them passes through the Center, and, when neither is bisected, by the other.



From the Center, F, draw FG and FH perpendicular to AB and CD; and join AF, FD, and EF.

Now AB is bisected in G, and cut unequally in E;
wherefore, AE × EB + EG □ = AG □ - - - 5.2.
Add, on both sides, FG □;
then, AE × EB + EG □ + FG □, = AG □ + FG □ - Ax. 6. 1.
But EF □ = EG □ + FG □; & AF □ = AG □ + FG □ - 20.
Therefore, AE × EB + EF □ = AF □.

After the same manner, it may be proved, that the Rectangle CED + EF □ is equal to AF □.

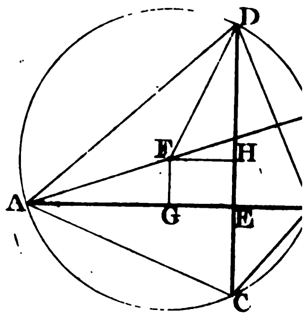
For, CD is also bisected, in H, and unequally cut, in E.
Wherefore, CE × ED + HE □ = HD □. Add HF □ to both.
Then, CE × ED + HE □ + HF □ (equal EF □) 20. 1
is equal to HD □ + HF □; i. e. equal FD □.
But FD = AF; wherefore, FD □ = AF □ - Ax. 1.
consequently, CE × ED + EF □ = AF □; (equal FD □)
But, it was proved, that, AE × EB + EF □ = AF □.
Wherefore, taking away EF □, from both,
there remains AE × EB = CE × ED. - - - - Ax. 7. 1.
Or, the Rectangle AEB, equal CED. Q. E. D.

THEO-

THEOREM XV.

If two Chord Lines, interfect at right angles; the four Squares, of the Segments of thofe Chords, will be equal to the Square of the Diameter.

And, the Squares of the two oppofite Sides of a Quadrilateral, formed by Right Lines, joining the extreme Points of the Chords, are alfo equal to the Square of the Diameter.



Let the Chords AB and CD cut each other perpendicularly, in E.

Then, the Squares of AE, EB, EC, and ED, are equal to the Square of AI; the Diameter of the Circle.

From the Center, F, draw FG and FH parallel to the Chords CD and AB; and join AF, and FD.

DEM. Now, fince AB & CD cut each other perpend. - Hyp.
and, FG, FH, are refpectively parallel to CD & AB - Con.
FG and FH are penpendicular to AB & CD. - C.2.4.1.
Then, AB is bifected, in G. (1.3.) and cut unequally, in E.
wherefore, $AE\square + EB\square = 2AG\square + 2GE\square$; - 9. 2.
alfo $CE\square + ED\square = 2EH\square + 2HD\square$ - - - fame
Wh. $AE+EB+CE+ED\square = 2AG + 2GE + 2EH + 2HD\square$
But GFHE is a Parallelogram, by Conftruction.
wherefore, $FH = GE$, and $FG = EH$ - - - - 15. 1.
Conf. the Squares of the four Segments, AE, EB, CE, & ED are eq. to the Squares of AG, GF, FH & HD, twice taken.
But, $AF\square = AG\square + GF\square$; $FD\square = FH\square + HD\square$ - 20.1
conf. $2AF\square + 2FD\square = 2AG + 2GF + 2FH + 2HD\square$
Wherefore, $AE + EB + CE + ED\square = 2AF\square + 2FD\square$;
i.e. $= 4\,AF\square$; for, AF is equal to FD.

But four times AF □ = AI □; i. e. of the Diameter. - 4.
Th. the Squares of the four Segments, AE, EB, CE, & ED, are equal to the Square of the Diameter.

2nd. Having joined the extremes AD, DB, AC, and CB. Then, the Squares of AD and CB, together, are equal to the Squares of AC and DB together.

For, AD □ = AE □ + ED □; and, CB □ = CE □ + EB □
But, AC □ = AE □ + EC □; and DB □ = DE □ + EB □
therefore, AD □ + CB □ = AC □ + DB □
consequently, each is equal to the Square of the Diameter.

This extraordinary property of the Circle is otherwise demon- strated by Mr. Stone; which, notwithstanding it is indisputabl true, does not carry with it that positive conviction; in so much that, a young Geometrician would be somewhat at a loss t perceive it.——It is as follows:

Because, the Angle AED, is a Right one, the Angles EAD, ADE, are equal to a Right Angle (32.1.) wherefore the Arks, AC, DB, on which they stand, are, together, equal to half the circumference of a Circle (26.3.)

Therefore, because the Angle in a Semicircle is a Right Angle, the Squares of AC and DB are equal to the Square of the Diameter (47.1.) And, because the Angles at E are Right Angles, the Squares of AE and EC are equal to the Square of AC, and the Squares of DE, EB, equal to the Square of DB. Therefore, the Squares of AE, EC, DE, & EB, are equal to the Squares of AC & DB; which, were proved equal to the Square of the Diameter. Therefore those four Squares, are equal to the Square of the Diameter.

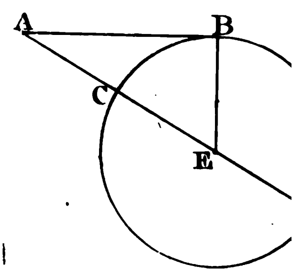
The References are, in this Demonstration, according to Euclid.

SHOL: *It is worthy of observation, that, as, in a Semicircle, the Sides containing the Right Angle are two Chords perpendicular to each other, and are equal to the Square of the Diameter; so the Squares of the Segments of all Chords which cut each other at right angles, are equal to the Square of the Diameter.*

THEO-

THEOREM XVI. 37. Euclid.

If, from any Point without a Circle, two Right Lines be drawn, the one a Tangent, the other a Secant (i. e. one touching the Circumference, the other cuting it twice) the Rectangle, under the whole Secant and the external Segment (between the Point, aſſumed, and the Circle) will be equal to the Square of the Tangent.



A is the aſſumed Point.

Let AB, touch the Circle, CBD, in the Point B; AD is a Secant, cuting the Circumference in two Points, C and D.

Then, the Rectangle DAC is equal to the Square of AB.

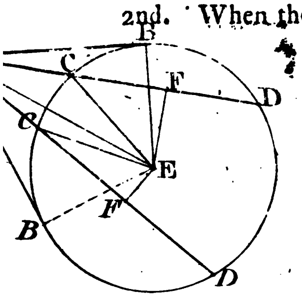
First, when AD paſſes through E, the Center of the Circle.

Draw EB, from the Center to the Point of contact.

DEM. Now, becauſe CD is biſected, in E, and CA is added to it; AD × AC + CE □ = AE □ - - 6. 2
and, AE □ = AB □ + BE □. - - - - - - 20. 1
For, ABE is a Right Angle - - - - C. 3. 8
wherefore, AD × AC + CE □ = AB □ + BE □.
But, BE = CE; conſequently BE □ = CE □. - Ax. B. 2
wherefore, AD × AC + CE □ = AB □ + CE □.
Take CE □ from both, there remains ▭ DAC = AB □.

 2nd. When

2nd. When the Secant falls on either side of the Center.



From the Center, E, draw EF perpendicular to the Secant, AD, or *AD*, and join the Points CE and EB.

Then, because EF is perpendicular to CD, CD is bisected in F. - - - C. 1.1

And, because CD is bisected, in F, and AC is added; $AD \times AC + CF\square = AF\square$ (6. 2.) Add EF $\square$ to both;
Then, $AD \times AC + CF\square + EF\square = AF\square + EF\square$ - Ax. 6.
But, $CF\square + EF\square = CE\square$;
and $AF\square + EF\square = AE\square$. - - - - Th. 20.
wherefore, $AD \times AC + CE\square = AE\square$;
i. e. equal to $AB\square + BE\square$. - - - - - - 20.
But, $BE = CE$; wh. $AD \times AC + BE\square = AB\square + BE\square$.
And, if $BE\square$ be taken from both,
there is left $AD \times AC = AB\square$. Q. E. D.

COR. Hence, it is evident, that the Rectangles under eve Right Line, cuting a Circle, from the same Point witho the Circle, and the external Segment, are equal.

The Rectangle $DAC = \mathit{DAC}$; for, they are each equ the Square of the Tangent, AB.

2nd. If from the same Point A, without a Circle, two Rig Lines AB, *AB*, are drawn, to the Circle, one on eac side, touching the Circumference, those two Tangen are equal.

For, $AB\square = \mathit{AB}\square$, being each equal to the Rect. DA
But, equal Squares have equal Sides;
consequently, AB is equal to *AB*. - - - Ax. B.

3rd.

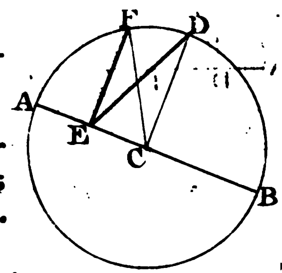
3rd. If from any Point, without a Circle, two Right Lines are drawn to the Circle; one of which Lines cuts the Circle, and the other meets the Circumference, in some Point only; and, if the Rectangle under the whole Secant and the external Segment, be equal to the Square of the other Line, that other Line is a Tangent to the Circle, and touches the Circumference in that Point.

THEOREM XVII.

If from any Point, in the Diameter of a Circle, there be drawn two Right Lines to the Circumference; one perpendicular to the Diameter, the other to the middle point of the Ark of the Semicircle; the Squares of those two Lines, together, will be equal to half the Square of the Diameter.

Let AB be a Diameter, C the Center, and D the middle Point of the Semicircle, ADB.

From the Point E, draw EF, perpendicular to AB, and join ED.



I say, the Squares of EF and ED, together, are equal to AC Square twice taken; i.e. to half the Square of the Diameter, AB.

Join CD and CF.

DEM. $CF\square = EF\square + EC\square$ - - - - - 20. I

Wherefore, $EF\square + EC\square = AC\square$; for $AC = CF$.

But $ED\square = CD\square + EC\square$; and $DC = CF = AC$.

Conf. $ED\square - EC\square = EF\square + EC\square$; i.e. $= AC\square$.

Wherefore $EF\square + ED\square = CF\square + CD\square$.

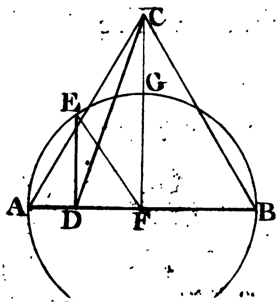
Therefore, $EF\square + ED\square = 2AC\square$; i. e. $= AB \times AC$.

THEO-

THEOREM XVIII.

If from the Vertex of an equilateral Triangle, constructed on the Diameter of a Circle, a Right Line be drawn cuting the Diameter; and, from that Point, another Line be drawn, perpendicular to the Diameter, cuting the Circumference; the Squares of those two Lines, together, will be equal to the Square of the Diameter.

From C, the Vertex of the equilateral Triangle ACB, on the Diameter AB, draw CD, at pleasure, cuting the Diameter in D; from which, draw DE perpendicular to AB.



I say, that the Squares of CD, DE, together, are equal to the Square of AB.

If CD be drawn perpendicular to AB, as CF, it is manifest.

For, CF □ + FG (equal AF) □ = AC (equal AB) □. - - - - - - 20. 1

Let CF be drawn, perpendicular to AB, and join EF.

Dem. Because CF is perpendicular to AB, and AB is equal to CB, AB is bisected in F; - - C.3.9 .1

wherefore, F is the Center of the Circle. - Def. 20 & 1

Now, AC = AB (Con) and AC □ = CF □ + AF □ - 20. 1

But, AF = EF, and EF □ = ED □ + DF □, eq. AF □ - same;

wherefore, AC □ = CF □ + DF □ + DE □.

But, CD □ = CF □ + DF □ - - - - - - 20. 1.

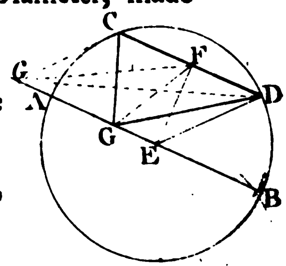
consequently, AC □ = CD □ + DE □.

But AB = AC (Con.) Therefore, AB □ = CD □ + DE □.

THEO-

THEOREM XIX.

If a Chord Line be parallel to a Diameter of a Circle, and, from each extreme of the Chord, Right Lines are drawn to any Point in the Diameter; the Squares of thoſe Lines, together, are equal to the Squares of the Segments of the Diameter, made by that Point.



Let the Chord CD be parallel to the Diameter, AB;
and, to any Point, G, in the Diameter, draw CG and DG.

I ſay, the Squares of CG, GD, together, are equal to the Squares of AG and GB. Let E be the Center.

Biſect CD, in F; and join FG, FE and ED.

DEM. Then, becauſe CD is biſected, in F,
$CG\square + GD\square = 2FG\square + 2FD\square$ - - - 17. 2
And, becauſe CD is biſected, EF is perpend. to CD. - 1.
But, CD is parallel to AB; conſ. EF is perpend. to AB - 4. 1
wherefore, $FG\square = EF\square + EG\square$;
and, $ED\square = EF\square + FD\square$. - - - - - - 20. 1
conſ. $CG\square + GD\square = 2GE + 2EF + 2FD$ Squares;
i. e. equal to $2GE\square + 2ED\square$. But $AE = ED$;
wherefore, $CG\square + GD\square = 2GE\square + 2AE$, equal ED, $\square$
But, $AG\square + GB\square = 2AE\square + 2GE\square$ - - - 9. 2
therefore, $CG\square + GD\square = AG\square + GB\square$. - Ax. 3. 1

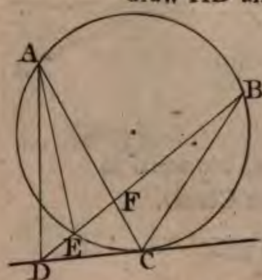
THEO-

THEOREM. XX.

If two Right Lines are drawn, from any two Point in the Circumference of a Circle, to the fam Point in a Tangent to that Circle; thofe Line will make the greateft Angle, when they meet i the Point of contact.

Let A and B be the Points affumed, in the Circumferenc of the Circle ABC.

Draw AC and BC to the Point C, in which a Tange DC, touches the Circle; and alfo, to any other Point, draw AD and BD.



I fay, the Angle ACB is greater than ADI

Join AE

DEM. The Angle AEB=ACB. - - Th. 10 ftanding on the fame Ark, AB.

But, the Angle AEB is greater than ADI for it is equal to ADB+DAE - 10.

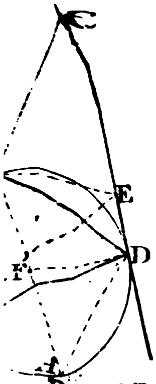
Therefore, ACB is greater than ADB.

COR. Hence it is evident, that an Angle, AFB, which fall within the Circle, is greater, and an Angle ADB withou the Circle, is lefs, than any Angle, AEB, touching th Circumference, and ftanding on the fame, or an equal Ark, AB.

From hence may be deduced the following Problem.

Any Right Line being given, and two Points given, o affumed, without the Line; to determine the Poi in that Line, to which, if two Right Lines be drawn from the given Points, they fhall contain a greate Angle than any other Right Lines drawn, fro thofe Points, to any other Point in that Line.

A an



A and B are the aſſumed Points; and, let CD be a given Right Line.

It is required to find the Point D, ſo, that, if two Right Lines AD and BD be drawn, to that Point, the Angle ADB, ſhall be greater than any other Angle, made by Right Lines drawn, from the ſame Points A and B, to any other Point in CD.

When the given Points, A and B, are not equi-diſtant from CD.

Draw AB, and produce it till it cuts DC, in C.

Find the Point D, ſo, that the Square of CD ſhall be equal to a Rectangle under the whole Line AC and the Segment BC. (by Prob. 25, or 30.)

Then is D the Point ſought; in which, two Right Lines, AD and BD, meeting, ſhall contain a greater Angle, than to any other point in CD.

DEM. For, becauſe the Rectangle ACB is equal to the Square of CD. (Con.) if a Circle, ABD, be deſcribed, through the Points A, B, & D. (by Pr. 40.) CD will be a Tangent to that Circle, and D the Point of contact-C. 3. 16. Therefore, ADB is larger than any other Angle, AEB, made by Right Lines, drawn from the ſame Points, A and B, to any other Point, E, in the given Right Line CD.

If the given Points, A and B, are equidiſtant from the given Line; a Right Line joining thoſe Points, being biſected, and a Perpendicular, to that Line, drawn, from the Point of biſection, cuting the given Line, will give the Point required.

For, a Right Line joining the Points will be parallel to the given Line, (Def. 7.) wherefore, a Line perpendicular to it is alſo perpendicular to the given Line (3. 1.) and will, conſequently, cut it in the Point of contact of a Circle deſcribed through that Point and the two given Points (C. 1. 1. and 2. 8. 3.)

ELEMENTS

OF

GEOMETRY.

BOOK IV.

THE fourth Book of Euclid's Elements is not properly elementary, but practical or mechanical; it treats of the inscription and circumscription of right-lined Figures in and about Circles. It is of use in Trigonometry, Astronomy, &c. and also, in Fortification or military Architecture, which seems to depend entirely on it.

As I do not think it proper to deviate from the order of Euclid, in his Books, I have therefore made it the fourth; otherwise, I should have given it a place amongst the other Problems, in practical Geometry, it being entirely problematical. I have given Demonstration of each Proposition, as brief as it will possibly admit of; because, I would not have the Work deficient in any part; which would be sufficient reason, with some Persons, to condem the whole. Nevertheless I am of Opinion, that great part of it does not require Demonstration; being sufficiently evident from the Construction; especially, as no other part of the Elements is dependant on it.

There are, in some Books extant, sundry other Propositions relative to the inscription and circumscription of right-lined Figures, in and about other right-lined Figures; particularly in Stone's Euclid; and in a curious Tract of practical Geometry by Le Clerc. But, as there is nothing of real utility in them, nor any thing extraordinary to recommend

them to the Curious, I ſhall not, by uſeleſs additions, to this, detain the Reader from matter of greater Importance in the fifth and ſixth Books. Indeed this may be paſſed over entirely, for the preſent, and proceed immediately to the fifth; there being nothing, in the fourth, neceſſary to be known previous thereto.

DEFINITIONS.

I have already, in the General Introduction, defined the Terms, to deſcribe, to inſcribe, and to circumſcribe; a repetition of which would be unneceſſary.

1. A Right-lined Figure is ſaid to be inſcribed, in a Circle, or to have a Circle circumſcribing it, when every Angle of the Figure, touches the Circumference of the circumſcribing Circle.

2. A Right-lined Figure is, then, ſaid to circumſcribe a Circle, or to have a Circle inſcribed, when every Side of the Figure touches the Circumference of the Circle.

3. A Right-lined Figure is ſaid to be inſcribed, or to circumſcribe a Right-lined Figure; when every Angle of the inſcribed one, touches every Side of the circumſcribing Figure.

4. So likewiſe, A Circle is ſaid to be inſcribed, in a Right-lined Figure, when it touches every Side of the Figure; or to circumſcribe a Right-lined Figure, when the Circumference touches every Angle.

N. B. By inſcribing any Figure within a Circle, or any Right-lined Figure within another, is underſtood, the deſcribing a Figure, like or ſimilar to the given one, the largeſt poſſible, to be contained in the other.

5. A Right Line is ſaid to be applied to a Circle, when each extreme is in the Circumference.

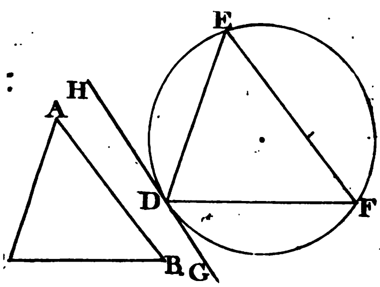
N. B. To inſcribe a Right Line in a Circle, is the firſt Problem of Euclid's Fourth Book, which I think entirel- unneceſſary, as a Problem, nothing more being required, i- the Operation, than to take the given Line, in the Compaſſe and placing one point in the Circumference, at pleaſure, cut the Circumference, with the other.

A Right Line, greater than a Diameter, cannot be inſcrib- in a Circle.

All the Propoſitions of this Book, follow in the or-e of Euclid, omitting the Firſt.

PROPOSITION II.

To inſcribe a Triangle in a given Circle, equiangula to a given Triangle.



Let ABC be the given Triangle, to be inſcribed in the Circle DEF.

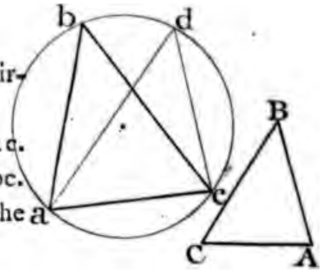
Draw at pleaſure the right line GH, touching the Circle in any point of its Circumference; as D.

Make the Angle GDF equal to A. i. e. to any Angle of the Triangle; and HDE equal to another, B, and join EF.

DEM. The Angle GDF (equal A) is equal to DEF;
and HDE (equal B) = DFE - - - - - - 13. 3
Wherefore, the Angle DEF, being eq. A, and DFE eq. B, the remaining Angle, EDF, muſt neceſſarily be equal to C.
Th. the Triangle DEF is equiangular to ABC. - 10. 1

N. B. By taking the Angles in this order, the Triangle inſcribed will be poſited as the given one; but any two being taken, the Triangle, inſcribed, will be the very ſame.

Otherwiſe;



Otherwiſe:

Draw at pleaſure a d, cuting the Circumference in a and d.

Make the Angle, a d c, equal to B, & join a c.

Make, c a b, equal to the Angle A, & join b c.

The Triangle a b c is equiangular to the Triangle ABC.

For, the Angle a b c = a d c (10. 3.) equal B, by Con.

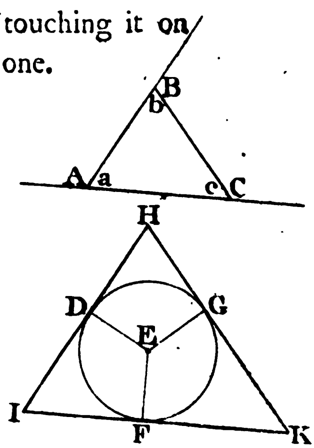
And, the Angle c a b was made equal to A;

Conſ. a c b is equal to the remaining Angle C. - 10. 1

Therefore, the Triangle, a b c, is equiangular to ABC.

PROPOSITION III.

To circumſcribe, that is, to deſcribe or draw, a Triangle, about a given Circle (touching it on every Side) equiangular to a given one.



Let ABC be the given Triangle, and DFG the Circle given.

Produce any Side of the Triangle, as AC, both ways; and draw a Radius, DE.

Make the Angle DEF, equal to the external Angle A; and EFG equal C.

Through the Points D, F, and G; draw Tangents to the Circle, cuting each other in H, I and K.

Then is H I K the Triangle required.

N.B. Two Angles, DEF, DEG, or FEG, being made equal to any two external Angles of the Triangle, will give the ſame Triangle, H I K, but differently poſited.

DEM.

DEM. Because the Right Lines, HI, IK, and HK, touches the Circle, in the Points D, F, and G; and, from those Points, the Lines DE, FE, and GE are drawn to the Center, the Angles, EDI, IFE, &c. are right ones. - 8. 3

And, because the Angles, D, E, F, and I, of every Quadrilateral, are equal to four Right Angles, - 11. 3

the Angle I + DEF = two Right ones.

But, the Angle DEF is equal to A, by Construction.

consequently, DIF is equal to BAC. - - 1. 1. & Ax. 7.

After the same Manner, the Angle K may be proved equal to ACB; and the Angle H, equal ABC.

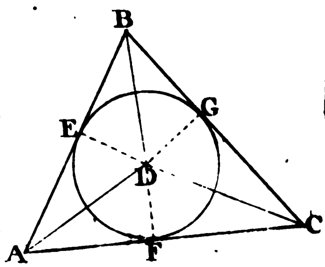
Therefore, the Triangle HIK is equiangular to ABC.

The Reason of this appears obvious, from the two Theorems deduced from the 10th Proposition, Book I. For, since (by the first) all the Angles of every Right-lined Figure are, together, equal to twice as many Right Angles as it has Sides, wanting four; and by the 2nd, all the external Angles are equal to four Right Angles. Also, by the 10th, the three Angles of every Triangle are equal to two Right Angles; and all the Angles, about a Point, are equal to four; (2.1.) consequently, since the three Angles DEF, DEG, and FEG, are respectively equal to the external Angles, A, B, and C, of the Triangle ABC; the Angles H, I, and K are, also, respectively equal to the internal Angles, a, b, and c, of the Triangle.

PRO.

PROPOSITION IV.

To inſcribe a Circle in a given Triangle, (ABC) touching every Side of the Triangle.



Biſect any two Angles of the Triangle, ABC, and CAB, by the Right Lines BD, and AD, cuting each other in D; from which Point, draw a Perpendicular (DE) to any Side (AB) of the Triangle.

With the radius DE, on the Center D, deſcribe the Circle EFG, which will touch every Side of the Triangle, ABC.

Draw DF & DG, perpendicular to the Sides, AC & BC.

DEM. Now, AF is equal to AE - - - - C. 2. 16. 3. and, the Angle EAD is equal to DAF. - - - Con. Wherefore, in the Triangles AED, AFD, the two Sides AE, AD, are reſpectively equal to the two Sides AF, AD; and they contain equal Angles.

Therefore DE is equal to DF. - - - - - 8. 1

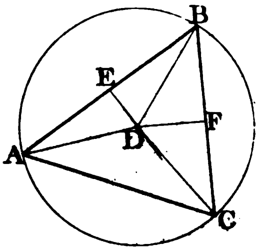
After the ſame manner, DG may he proved equal to DE, and alſo to DF.

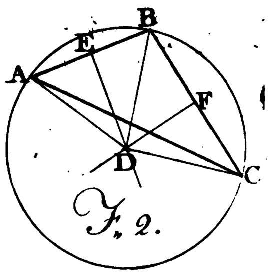
Wherefore, the three Lines DE, DF, and DG, being perpendicular to the three Sides of the Triangle, and being proved equal among themſelves, the Circle, EFG, will paſs through the three point E, F, and G; and, conſequently, will touch every Side of the Triangle ABC.

PRO-

PROPOSITION V.

To circumscribe a given Triangle with a Circle; or, to describe a Circle about a given Triangle.





ABC is the Triangle given.

Bisect any two Sides of the Triangle, AB and BC, in E and F.

Draw ED and FD, perpendicular to AB and BC, cuting each other in D the Center of the circumscribing Circle.

If the Triangle has a Right Angle, the Hypothenuse bisected gives the Center.

If an Angle be obtuse, the Center of a circumscribing Circle falls without the Triangle; the Operation is the same in both, (see Fig. 2.)

Join the Points AD, BD and CD.

Dem. Because the Side AB is besected in E, and DE perpend. to AB, the Sides AE, ED, of the Triangle AED are equal, respectively, to BE, ED. of the Triangle BED and the Angle AED is equal to BED. - - Def. 1
wherefore, AD is equal to BD - - - - - 9.
for they subtend equal Angles, in the Triangle ABD.

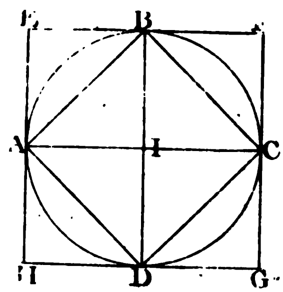
In the same manner may be proved, CD=AD, or BD. Consequently, D is the Center of the circumscribing Circle, ABC.

PRO-

PROPOSITION VI.

To inſcribe a Square in a given Circle; and to deſcribe a Square about a Circle. ABCD and EFGH.

Draw two Diameters, AC and BD, at Right Angles. Join the Extremes, A, B, C, D, and a Square is inſcribed.



2nd. Draw Tangents through the extreme Points of the Diameters, meeting in E, F, G, and H; or draw Lines, touching the Circle, parallel to each Side of the inſcribed Square, and EFGH is a Square circumſcribing the Circle.

DEM. The Diameters AC & BD biſect each other, - 16. 1
therefore, AI, IC, IB, and ID, are equal;
and they contain equal Angels, AIB=BIC &c. - - Con.
Wh. AB, BC, CD & AD, are all equal amongſt themſelves.
And, the Angles ABC, BCD, &c being in a Semicircle, are Right Angles. Therefore ABCD is a Square.

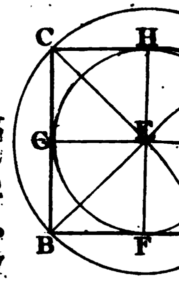
2ndly. Becauſe EF touches the Circle, in the Point B, it is perpendicular to the Diameter BD. - - - 8. 3
And, becauſe HG is perpend. to BD, HG is parallel to EF;
For the ſame reaſon, EH is parallel to EG;
Therefore, EFGH is a Parallelogram. - - - Def. 33.

But, AC is perpendicular to BD. (Con.) wherefore, EH is perpendicular to HG, and ſo is EF to FG and to EH; conſequently, EFGH is a Rectangle. - - - Def. 34
But EF, FG, EH, and HG, are each equal to AC, equal BD.
Th. EFGH is equilateral; conſ. it is a Square - Def 35

N. B. The Square, EFGH, circumſcribing a Circle, is double the Square, ABCD, inſcribed in the ſame Circle.
For the Triangle ABC = half the Rectangle AEFB - - 17. 1
And ADC is equal to half the Rectangle AG;
conſequently, the Square EFGH is double ABCD.

PROPOSITION VII.

To inſcribe a Circle in a Square; and to circumſcrib
i. e. to deſcribe a Circle about a Square.



Let ABCD be a given Square

Draw the Diameters AC & BD cuting each other in E.
From the Center E, draw EF perp. to AB.
With the Radius EF, on the Center E, deſcribe a Circle, which will touch every Side of the Square ABCD.

Draw EG perpend to BC; produce GE to I, and FE to H.

DEM. Now, becauſe ABCD is a Parallelogram, the Diagonals AC & BD biſect each other, in E; - - - 16. 1
And becauſe AB, BC, are reſpectively equal to AB, AD, and contain equal Angles; AC=BD - - - - 8. 1
conſequently their halves are equal. - - - Ax. 1.

Then, ſince AE, EB, and EC are equal, the Triangl AEB, BEC are Iſoſceles; wh. AB and BC are biſected b the Perpendiculars EF & EG, conſ. FB=BG. - C. 4. 9.

But, ABC is a Right Angle, and AB=BC. - Def. 3 and ſince AC is biſected in E, EB is perpendicular to AC Wh. AEB being a Right Angle ABE, EAB, are half Righ ; conſequently, BE biſects the Right Angle ABC.

Now ſince in the Triangles EFB, BGE, the Side, FB, BE, are reſpectively equal to GB, BE; and the Ang e FBE=EBG, the remaining Side EF=EG.

After the ſame manner EH and EI may be proved equ l to EF and FG; conſ. the Circle FGH touches every Si e of the Square.

2ndly

2ndly. Having drawn the Diameters of the Square AC & BD; with the Radius EA or EB, defcribe the Circle, ABCD.

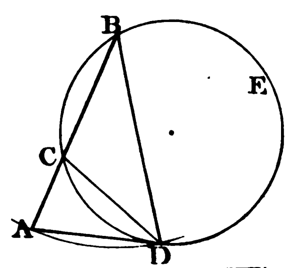
This needs no further Demonftration.
For, fince the Diagonals of a Square are equal, and mutually bifect each other, (16. 1.) the halves EA, EB, EC, and ED are alfo equal; wherefore, a Circle, ABC, will pafs through every Angle of the Square.

N. B. A Circle (FGHI) infcribed in a Square, is equal to half a Circle (ABCD) circumfcribing that Square.

For, the Square of AC=AB □ +BC □ ; confequently, AC □ is double BC, equal FH, □.
But the Areas of Circles are, to each other, as the Squares of their Diameters, - - - - - - - - - - C. 1. 14. 6.
therefore, the Circle ABCD is double FGHD.

PRPOSITION VIII.

To make an Ifofceles Triangle, having its Angles, at the Bafe, each double the Angle at the Vertex.



Let AB be any given Right Line for one of the equal Sides.
Divide AB in extreme and mean Proportion, in C. (Prob. 35.)
With the Radius AB, on B, defcribe an Ark, AD.
Make AD equal to CB, and draw BD.

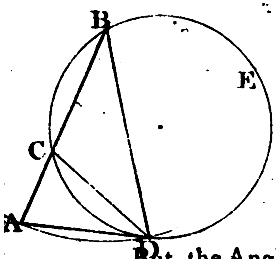
I fay; in the Triangle ABD, the Angles A and D, at the Bafe, are each double the Angle B, at the Vertex.

Join C D, and defcribe a Circle through B, C, and D, (Prob. 40.) or, by the 5th of this.

DEM. Becaufe the Rectangle BAC=BC□ - - - 11. 2

And AD=BC (Con.) the Rectangle BAC=AD□;

Wherefore, AD touches the Circle BDE, at D - 16. 3



(For A is a Point without, that Circl
from which, two Right Lines are draw[illegible]
one (AB) cuting, the other, (AD) touch[illegible]
the Circle;)

Therefore, the Angle ABD=ADC. - 13. 3

(For, CD cuts the Circle BDE, in t[illegible]e
Point of contact, D.)

But, the Angle ACD=ABD+CDB, i.e. =ADC+CD[illegible];

And, the Angle BAD=ADB (9.1) for AB=BD. - Cor[illegible] 1.

consequently, BAD=ACD; wh. CD=AD - C. 3. 9. 1

But, AD=CB (Con.) wherefore, CD=CB; - Ax. 3. 1

consequently, the Angle CDB=CBD - - Th. 9. 1

But, the Angle ADC=CBD; (proved, above)

wherefore, ADC is equal to CDB. - - - Ax. 3. 1

and therefore, ADB, BAD are each double ABD.

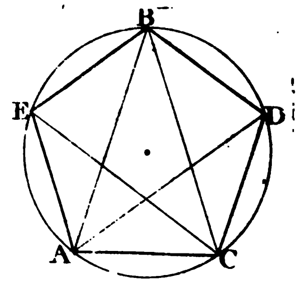
The Conftruction of this particular fpecies of Triangles, in refpect of itfelf, is of little or no ufe; or, I fhould have given it a place, amongft the other Problems, in Practical Geometry. But, the infcribing a regular Pentagon in a Circle, according to Euclid, in the next Propofition, depends on it entirely; but, which may, for real ufe, in practice, be more readily conftructed. Neverthelefs, for the manner and elegance of its Demonftration, I did not think proper to omit it.

PRO

PROPOSITION IX.

To inscribe a regular Pentagon in a Circle.

ABC is the given Circle.



Inscribe an Isosceles Triangle, ABC, whose Angles BAC, ACB are, each, double the Angle ABC - - Prop. 1

Bisect each of the Angles BAC, ACB by the Right Lines AD and CE, cuting the Circumference in D and E.

Join the Points, A & E &c. by the Right Lines AE, EB, BD, and DC.

AEBDC is a regular Pentagon.

DEM. Now, because each Angle, BAC, ACB, is double the Angle ABC; (Con.) and those Angles are bisected, the Angles, ABC, ACE, ECB, BAD, and DAC, are equal to one another.

But equal Angles stand on equal Arks, - - C. 2. 9. 3 and, equal Arks have equal Chords or Subtenses, - 2. 3. 3 wherefore, the Right Lines AC, AE, EB, &c. are all equal; and consequently, the Pentagon, AEBDC, is equilateral.

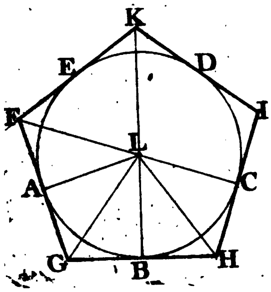
I say, it is also equiangular.

For, because the Arks AE, EB, &c. are all equal, the Angles ACE, ECB, &c. are equal.
wh. the Angle EBA + ABC + CBD = EAB + BAD + DAC.
consequently the Angle EBD = EAC; and so, of all the rest.

Or, because each Angle of the Pentagon, AEB, EBD, &c. stands on equal Arks, ACDB, FACD, &c; (for each is triple the Ark, AC.)
Therefore, the Pentagon AEBDC is equiangular.

PROPOSITION X.

To defcribe a regular Pentagon about a Circle.



Let ABCDE be the angular Points of a Pentagon, infcribed in a Circle, through which draw the Tangents FG, GH, HI, &c. cuting each other, in F, G, H, I, and K, i. e. having, made the Arks AB, BC, &c. equal, (viz. each equal 72 Degrees) through the Points A, B, C, &c. draw the Tangents, as before.

The Pentagon FGHIK is equilateral and equiangular.

Let L be the Center of the Circle; draw AL, BL, &c. alfo draw FL, GL, &c.

DEM. AL, BL, &c. are perpendicular to FG, GH, &c. - 8 3
and they are all equal between themfelves - - - Ax. 1. 3
The Angle ALB is equal to BLC, by Conftruction.
AG is equal to GB, and BH equal to HC - - C.2.16.3
wh. the Triangles ALG, GLB, BLH, &c. are equilateral and equiangular to each other; for GB=BH, &c. - 11.1
AF=AG, and CI=CH, (fame) conf. FG=GH, - Ax. 1
After the fame manner, FK and IK, may be proved equal between themfelves, and alfo to FG, &c;
therefore, FGHIK is equilateral.

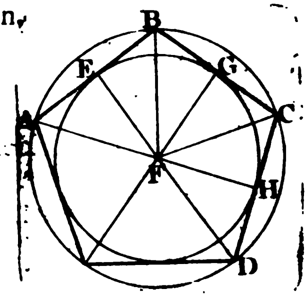
And, fince the Triangles ALG, GLB, &c. are equiangular, the Pentagon is alfo equiangular.
For, the Angles AGL, LGB, alfo BHL and LHC are equal amongft themfelves; confequently, FGH=GHI.

After the fame manner, the Angles, I and K, may alfo be proved equal betwen themfelves, and alfo to F, G, & H.
Th. FGHIK is a regular Pentagon, circumfcribing a Circle.

Note. If the Circumference of a Circle be divided in five equal Parts, and the Points joined by Right Lines; then if Tangents to the Circle be drawn, parallel to every Chord, a regular Pentagon will be circumfcribed.

PROPOSITION XI.

To inscribe a Circle in a regular Pentagon; and, to describe a Circle about a Pentagon.



Bisect any two Sides, AB, and BC, of the Pentagon ABCD, by the Perpendiculars EF and FG, intersecting in F; on which, with the Radius EF or FG, describe a Circle; which, will touch every Side of the Pentagon.

2nd. With the Radius AF or FB (the Point F being found as before, or by bisecting two Angles, ABC and BCD) describe a Circle, which will pass through every Angle of the Pentagon.

DEM. Because AB $=$ BC, EB $=$ BG. Draw AF, FB, &c.
Then, in the Triangles EBF, FBG; FB $\square =$ FE $\square +$ EB $\square$.
and, FB $\square =$ FG $\square +$ BG $\square$; - - - - - - 20. 1.
consequently, FE $\square +$ EB $\square =$ FG $\square +$ BG $\square$. - Ax. 3. 1
But, EB $=$ BG (Ax. 1. 1.) wherefore, EB $\square =$ BG $\square$;
consequently, EF $\square =$ FG $\square$; and, therefore EF $=$ FB.
In the same manner, FH may be proved equal to FG;
F is, therefore, the Center of a Circle inscribed.

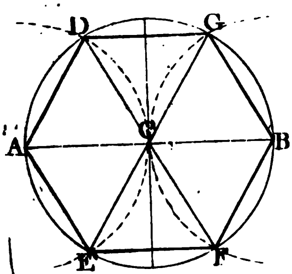
2nd. Because AB is bisected, in E, AE $=$ EB, & AE $\square =$ EB $\square$
Add EF $\square$, to both; and AE $\square +$ EF $\square =$ EB $\square +$ EF $\square$.
But, EF is perpendicular to AB, by Construction;
consequently, AEF, FEB are Right Angles. - Def. 10
wh. AF $\square =$ FB $\square$; and FB $\square =$ EB $\square +$ EF $\square$.
Consequently, AF $\square =$ FB $\square$; and therefore, AF $=$ FB.
In the same manner, FC, FD, &c. may be proved equal to AF and FB.
consequently, F is the Center of a circumscribing Circle.

PRO-

PROPOSITION XII.

To inſcribe a regular Hexagon in a given Circle.

In the given Circle, A G F, draw a Diameter, AB.



With the Radius of the Circle, AC, on the Centers A and B, deſcribe two Arks, DCE, FCG, cuting the Circumference in the Points D and E, F and G.

Or, having drawn the Ark DCE, only, cuting the Circle in D and E, draw DC & EC. and produce them to F and G.

Join the Points A and D, AE, EF, &c. by Right Lines, which compleats the Hexagon, ADGBFE. Q. E. F.

I ſay, it is both equilateral and equiangular.

DEM. Becauſe C is the Center of the Circle AGF, the Lines DC, AC, and CE are equal; and, becauſe A is the Center of the Circle DCE, AD, AC and AE, are equal; wherefore, ADC & ACE, are equal & equilateral Triangles.

But, the Angles of an equilateral Triangle are, each, equal to one third Part of two Right Angles. - C. 10.1 (becauſe, the Angles which are oppoſite to equal Sides are equal; and the three Angles of every Triangle are, together, equal to two Right Angles. - - 9 and 10.1) conſequently, DCA+ACE= two thirds of two R. Angles.

Now, becauſe the Right Line EC falls on the R. Line DF, the Angle ECF+ECD= two Right Angles - 1.1 and, becauſe DCA+ACE (eq. DCE)=two thirds of two Right Angles; ECF is equal one third part of two Right Angles; i. e. equal to ACE, equal ACD.

Again

Again; because the Right Lines AB, DF, & EG cut one another, in the Point C, the Angles DCG, GCB, & BCF are, respectively, equal to ECF, ACE, and ACD - 2. 1 Wh. those six Angles are all equal amongst themselves; conf. they stand on equal Arks; and conf. on equal Chords. Therefore, the Sides, AD, DG, GB, &c. of the Hexagon, are equal to one another.

2nd. Because the Triangles, ADC, CDG, &c. are equilateral, and consequently equiangular, - - - - - C. 2. 9. 1.
the Angle ADC + CDG = DGC + CGB, - - Ax. 3.
i. e. the Angle ADG = DGB; and so of all the rest.
Therefore, the Hexagon ADGBFE, inscribed in a Circle, is equiangular; and also equilateral.

Cor. The Side of a regular Hexagon, inscribed in a Circle, is equal to the Radius, or Semidiameter of the Circle.

For, the Triangles, ADC, &c. are proved to be equilateral.

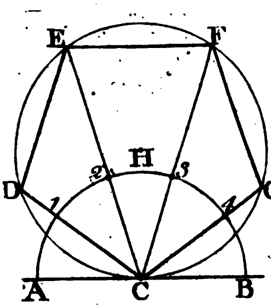
Hence, a Duodecagon may be inscribed in a Circle, by bisecting each Ark on the Side of a Hexagon inscribed. i. e. The Radius of the Circle being applied six times in the Circumference, each Ark bisected gives the Sides of a Decagon.

N. B. Any Poligon may be circumscribed about a Circle, having first inscribed one (of the Species required) by drawing Tangents, through every Angle, or parallel to every Side, of the inscribed Figure.

In Prob. 52, Practical Geometry, is given a general method for inscribing Poligons of every Denomination in Circles; of which Method, there is no Demonstration extant, having consulted some able Mathematicians concerning it.

I ſhall, here, give another general method, which admits of Demonſtration (granting an Ark of a Circle to be divided into any number of equal Parts) which is as follows.

Let DFC be a given Circle, in which it is required to deſcribe a regular Pentagon.



Draw AB, at pleaſure, touching the Circle DFC in any Point, C; on which Center, draw a Semicircle AHB, with any Radius at diſcretion.

Divide the Semi-circumference into five equal Parts, i. e. make the Arks A 1, 1 2, 2 3, &c. each equal 36 Degrees.

Draw C 1, C 2, and produce them till they cut the Circumference, in D, E, F, and G.

Join DE, EF and FG; and it is done.

DEM. Becauſe the Angs DCE, ECF, &FCG are equal - Con. the Arks, and conſ. the Chords DE, EF, &FG, are equal, Alſo, the Angle ACD (eq. DCE by Con.)=DEC. - 13.3 wh. the Triangle CDE is Iſoſceles; and CD=DE. - 9.1

After the ſame manner CG may be proved equal to FG. conſequently DC, DE, EF, FG, and GC are all equal. Therefore, the Pentagon DEFGC is equilateral; and, being inſcribed in a Circle, it is neceſſarily equiangular, each Angle, CDE, DEF, &c. ſtanding on equal Parts of the Circumference, CGFE=DCGF.

This method of inſcribing Poligons is not in Euclid, nor in any Author I am acquainted with. It is neither ſo operoſe in the Conſtruction, and is more briefly demonſtrated.

Euclids method (for a Pentagon) of inſcribing a Triangle ſimilar to another, (which muſt firſt be formed) whoſe Angles at the Baſe are double of the Vertex, is very ingenious and perfectly Geometrical; but it is liable to great error in the Conſtruction; inſomuch, that it is extremely difficnlt to do it, with accuracy, and tedious in the Operation; whereas, this is both more expeditious and more to be depended on.

After

After the fame manner, a Heptagon, or any Poligon whatever, may be readily conftructed, as follows.

Draw AB, at pleafure, touching a Circle in C; on which Center defcribe a Semicircle, cuting the Tangent in A and B.

Divide the Ark of the Semicircle into as many equal Parts, as the Poligon, required, has Sides, viz. Seven (for a Heptagon) and, through every Divifion, draw C1, C2, &c. cuting the given Circle in D, E, F, &c; which will divide the Circumference into feven equal Parts, in thofe Points; and, being joined by Right Lines, compleats the Heptagon CDEFGHI.

The Demonftration is the fame as in the preceding Propofition. But, it may be further obferved, that, becaufe the equal Angles ACD, DCE, ECF, &c. are at the Center of the Circle AKB, and at the Circumference of the Circle EHC; and, becaufe the Angle at the Center of a Circle, on any Ark, is equal to an Angle at the Circumference, on twice that Ark (C.1. 9. 3.) therefore, the Semi-circumference, of one and the whole Circumference, of the other, are divided into the fame number of equal Parts, by the Right Lines, CD, CE, &c.

If AC be taken equal to the Radius of the given Circle, it is manifeft.

It may be alledged, that the whole Circumference may as eafily be divided into any number of equal Parts, as a Semi-circumference. 'Tis true it may, but that admits of no other Demonftration, the Sides being all equal by Conftruction; and is in no wife geometrical.

On account of the fingularity of this method, having never feen it in any other work, and becaufe I find it ufeful in Perfpective, in fome Cafes, I thought proper to infert it.

The Side of any Poligon inscribed in a Circle of an Radius is readily obtained, by making an Angle at th Center, of the number of Degrees, which is the Produ arising from the Division of 360 by the Denomination the Poligon. e. g. 360 divided by 5 (for a Pentago gives 72; for a Hexagon, 60; for an Octagon, 45; a Nonagon, 40; for a Decagon, 36. A Heptagon do not divide the Number of Degrees exactly; but is near, $51\frac{1}{2}$, or more nearly, 51, 43.

Wherefore, having (by means of a Protractor, or Lin of Chords) made an Angle, ACB, at the Center of Circle, equal 40 Degrees; the Chord AB, of the Ark stands on, is the Side of a Nonagon inscribed in th Circle FKG; i. e. AB is one ninth Part of th whole Circumference.

For, let the Angle ACB be constructe or measured on any Ark whatever, as al either of a greater or less Radius, the portio AB, of the Circle AKB, will be the sam for AB and ab are each one ninth part their respective Circumferences.

Hence may be deduced, an universal method for Con structing Poligons of any number of Sides, on a give Right Line, viz. by finding, what Angle the Side of t Poligon, subtends at the Center, as above; which in Nonagon is 40 Deg. Subtract 40, from 180, two Rig Angles; the remainder is 140, half of which is 70.

Construct an Isosceles Triangle, ACB, on the giv Line AB, whose Angles, at the Base, A and B, are ea of 70 Degrees; the Angle at the Vertex, C, will be 40.

C is the Center of a Circle which will circumscrib Nonagon, the measure of whose Sides is AB.

W

With the Radius CA or CB deſcribe a Circle, AKB, it will contain AB nine times, applied to the Circumference. For, if the Angle A C B, of 40 Degrees, be repeated 9 times, as in the Figure; it will include all the ſpace of the Circle, equal four Right Angles; and each Angle will be ſubtended by a Side of the Poligon; which is evident from the Figure.

The three Angles of every Triangle are, together, equal to two Right Angles, i. e. equal to 180 Degrees. (10. 1.) And, ſince the Angle C, at the Center of the Triangle ACB, is equal to 40 Degrees, conſequently, the two remaining Angles, CAB, CBA, at the Baſe, being equal (9. 1.) they are, each, equal to half the difference between the Angle ACB, and two Right Angles; i. e. of 70 Degrees, each, as by Conſtruction.

Otherwiſe. The external Angle of every regular Polygon is equal to an Angle at the Center, ſubtended by a Side.

For, all the external Angles of every Right-lined Figure are equal to four Right Angles (Th. 2. 10. 1.) and ſo are all the Angles about the Center of a Circle. - - - - C. 2. 2.1

Wherefore, an ordinate or regular Poligon, having all its internal Angles equal, will, likewiſe, have all its external Angles equal; and, being equal in number to the Angles at the Center, ſubtended by the Sides, each external Angle is, conſequently, equal to an Angle at the Center, ſubtended by a Side, e. g.

If the Poligon has eight Sides (an Octagon) the external Angle is the 8th part of 360, i. e. 45 Degrees; if a Nonagon, it is 40 Degrees; if a Decagon 36; each being equal to the Angle at the Center, ſubtended by a Side. Hence.

Let, AB be a Side given for a Nonagon; let it be produced both ways, towards D and E.

At the extremes, A and B, make the Angles DAF, EBG, each equal 40 Deg. and make AF and BG each equal AB.

Biſect

Bisect AF and BG, in H and I, and draw HC and IC perpendicular to AF and BG, intersecting in C; which, is the Center of a circumscribing Circle, as before.

It is evident, that the external Angle DAF (equal EBG) is equal to the Angle ACB at the Center.

For, ACB, BCG, and ACF, are congruous Isosceles Triangles; whose Angles at the Base, AB and BG, are 70 Degrees each; two, of which, form an internal Angle of the Nonagon; as ABC + CBG = ABG; and, the external Angle, EBG, is the Compliment of two Right Angles; consequently, it is equal to the remaining Angle ACB, of the Triangle ACB, or BCG.

Hence it is also evident; that, if any Right Line, as DE, cuts a Circle, and if from either Point, A or B, of the part *AB*, within the Circle, another Chord, AF or BG, be drawn, equal AB; the external Angle, DAF or EBG, made by that Chord and the Line DE, is equal to the Angle at the Center of the Circle, subtended by the Chord, AF or BG (equal AB).

From what I have advanced, is deduced the Table, after Prob. 47, Practical Geometry, for constructing Poligons; by dividing the Ark of a Quadrant, or Right Angle, into as many equal parts as the Poligon has Sides.

For, the Ark of a Right Angle being divided into five equal parts, each will be 18 Deg. wherefore, the Angle of a Pentagon being six of those parts, or one added to a Right Angle, i. e. $18 + 90 = 108$; which being subtracted from two Right Angles, or 180 Degrees, the Difference is 72, equal four times 18; which is the Compliment of two Right Angles; consequently, the external Angle of a Pentagon is equal to an Angle at the Center, subtended by a Side.

For a Nonagon, the Ark of a Right Angle is divided into 9 equal parts, each equal 10 Degrees.

Now, the external Angle being 40 Degrees, equal to the Angle at the Center, the Angle of the Nonagon is $50 + 90$ or

fiv-

five ninth parts added to a Right Angle; for, an internal and external Angle are, together, equal to two Right Angles - 1. 1

Therefore a Right Angle is to the Angle of a Nonagon as 9 to 14; difference 5, as in the Table.

The reason of all this is so very obvious, it is almost needless to say more about it; seeing it is manifest, that all the external Angles of any Right-lined Figure, whatever, are, together, equal to all the Angles about a Point, subtended by the Sides. i. e. equal to four Right Angles. And, all the internal Angles of any Right-lined Figure are equal to twice as many Right Angles as the Figure has Sides, wanting four, (Th. 1. 10. 1.) consequently, the external Angles being equal to those four (Th. 2. of the same) are equal to all the Angles at the Center; and, being equal, in number, they are equal in quantity.

Hence, the Angle of any regular Poligon, whatever, may be readily obtained.

For, in a Pentagon, all the internal Angles, together, are equal to six Right Angles; consequently, each Angle is 6 fifths of a Right Angle, i. e. it is equal to one Right Angle and one fifth part of a Right Angle; seeing, there are five Angles in the Figure, and they are, altogether, equal to six Right Angles.

The Angles of a Hexagon are, altogether, equal to eight Right Angles; consequently, each Angle is equal to eight sixths of a Right Angle, or four thirds; i. e. equal to one Right Angle and one third part of a Right Angle.

A Heptagon has all its Angles, together, equal to ten Right Angles; wherefore, each Angle is equal to ten sevenths of a Right Angle, i. e. equal to a Right Angle and three seventh parts of a Right Angle.

All the Angles of an Octagon being equal to twelve Right Angles, it is evident that each Angle is equal to one Right Angle and a half; i. e. to 12 eights, i. e. to six fourths, or three seconds, i. e. one and a half.

To particularize more would be unnecessary; as it is easy, from what I have said, to calculate the quantity of the Angle of any Poligon whatever, by the Proportion it bears to a Right Angle, or to two Right Angles; which must ever be more than the Angle of any Polygon whatever.

For Poligons which have an even number of Sides, it may be observed, that the division of a Right Angle, may be reduced to a lower Denomination; as for a Duodecagon, for instance; all its Angles, together, being equal to 20 twelfths = i. e. 10 sixths, or 5 third parts, i. e. to one Right Angle and two thirds; the Right Angle need be divided into three equal Parts, only, instead of twelve.

Whereas, those which have an odd number of Sides, cannot be reduced lower; but, in forming them, a Right Angle must necessarily be divided into as many parts as the Figure has Sides.

The method of inscribing a Quindecagon in a Circle, according to Euclid, is a matter of mere curiosity; first, to inscribe an equilateral Triangle, and afterward a Pentagon in the same Circle, in order to get a fifteenth part of the Circumference; which, notwithstanding it admits of perfect Demonstration, is liable to great error in Practice.

PROPOSITION XIII.

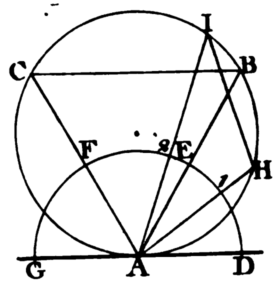
To find the Side of a Quindecagon inscribed in a Circle.

Let ABC be a given Circle, in which it is required to inscribe a Quindecagon.

Draw, DG, at pleasure, touching the Circle ABC in A. With any Radius, AD, on A, the Point of contact, describe a Seme-circle, DEFG.

Inscribe the equilateral Triangle ABC; by making the Arks DE, EF and FG, each equal to the Radius AD, and drawing AE, AF to B and C, and joining the Points B and C.

DEM.



DEM. For, the Arks, DE, EF, & FG, are equal by Conſtruction; wherefore, the Angles DAE, EAF, and FAG, are equal. But, the Angle BCA is equal to BAD, i.e. EAD; and, CBA is eq. to CAG - 13.3. conſ. BCA, CBA, BAC, being equal, reſpectively, to DAE, EAF, and FAG; the Triangle ABC is equiangular; and therefore it is equilateral - - C. 3. 9. 1.

Now, if the Ark of the Semicircle, DEFG, be divided into five equal parts, for Inſcribing a Pentagon, as, before into three (in E and F) and, if A1, A2 be drawn, cuting the Circumference, at H and I; AH and HI are Sides of a Pentagon, inſcribed.

In reſpect of the Operation, I leave it to the diſcretion of the Practitioner, to take what method he moſt approves.

Granting AB to be the Side of an equilateral Triangle, and AH, HI, to be the Sides of a Pentagon inſcribed; BI is the Side of a Quindecagon, or Poligon of 15 Sides.

For, becauſe AB is the Side of an equilateral Triangle, the Ark AHB is one third part of the whole Circumference.

Alſo, AH and HI being Sides of a Pentagon, the Arks AH and HBI, are each equal to one fifth part of the Circumference, i. e. the Ark AHI is equal to two fifths, equal ſix fifteenth parts; for HI, one fifth, isequal to three fifteenths.

Now, ſince AHB is one third part of the Circumference, it contains five fifteenths; for three times five is fifteen.

Wherefore, BI is equal to one fifteenth part, and is, therefore, the Side of a Quindecagon, inſcribed in the Circle ABC.

ELEMENTS OF GEOMETRY.

BOOK V.

OF PROPORTION.

THE fifth Book of the Elements of Euclid contains the sublime Doctrine of Proportion; which is, psrhaps, the moſt ſubtle manner of reaſoning, the moſt brief, ſolid and convincing, that the Art of Man could deviſe. It is eſſentially neceſſary in demonſtrating all the Propoſitions of the ſixth Book, which alone is ſufficient to recommend it. But, where we conſider its general and extenſive uſe, throughout the Mathematics, it is impoſſible to be diſpenſed with. The Doctrine it contains is not only uſeful in, but, it is the Criterion of almoſt every mathematical Science; inſomuch, that, without the knowledge it teaches, we ſhould not be able to advance one ſtep further.

I muſt ever be of opinion that the readieſt and eaſieſt way of acquiring knowledge, is the beſt. The Doctrine of Proportion is, in ſome meaſure, born with us; it is a portion of Reaſon, implanted in us by Nature; which does not require a formal Demonſtration, ſo much as barely to be illuſtrated, by ſome familiar Examples; and which, indeed may paſs for Axioms, for moſt of them are really ſuch.

In expreſſing and comparing Quantities, of any Species, it has been found neceſſary to make uſe of certain Symbols or Figures. Which Symbols, Marks or Characters, are entirely arbitrary, and might as well have been applied to ſignify the Qualities of Things, as Quantity. But, having been accuſtomed, from our infancy, to expreſs our Ideas of Proportion by Numbers; we no ſooner become acquainted with Numbers and the Characters applied to them, but we find it almoſt impoſſible, in our Ideas of Magnitude, or Quantity of any kind, to compare or eſtimate their Ratio, abſtracted from Numbers. In reſpect of Magnitude, as to ſolid Extenſion, or Body; we ſay, when compared, it is twice, thrice, or four times as big as another; ſo likewiſe of Extenſion, as to length, ſimply, we expreſs it by one, two, or three Yards, Chains, or Miles, &c; and ſo of any other Quantity whatever.

Hence, the mode of expreſſing Quantity, by Numbers, is the moſt natural, ſimple, and eaſy; and therefore, it is no wonder, when we find it expreſſed by other Characters, or by Lines, as it is done by Euclid and all his Commentators, in the fifth Book, that it appears, at firſt, to thoſe who have not conſidered it, quite foreign to the purpoſe; but, in reality, the difference between Lines and Numbers, in expreſſing Quantity, conſiſts, only, in being more familiarized to one than the other.

Therefore, ſince Lines, Characters, and Numbers, are only different modes of expreſſing Quantity, I muſt needs think that the beſt, which is the moſt natural and eaſy to conceive; beſides it is much more convincing; for, unleſs Lines are divided into the ſame equal Parts, whereby we may form a judgment by how much one Magnitude, or Quantity of any kind, exceeds another, we have no Idea of their Proportion; otherwiſe, than

that one does exceed or is more than the other; whereas, Numbers aſcertains the Ratio, whether it be equal or unequal.

If Characters (as A, B, C, &c.) are made uſe of, they only denote Quantity, ſimply, but in what Ratio, or Proportion, is beſt expreſſed by Numbers; algebraic Characters are not ſo expreſſive to thoſe who are not well verſed in them; they do not convey an Idea adequate to that given by Numbers, neither do they aſcertain the Ratio: and, if Lines are divided into equal Parts, to expreſs the Ratio, it is the ſame as Numbers; becauſe, we only know the Proportion by numbering the Parts.

It is eaſily known that there is Analogy, or ſimilarity of Proportion, when, in four Quantities, the firſt (i. e. any one) contains the ſecond (any other) or is contained, the ſame number of times, as a third contains the fourth, or is contained by it; whether it can be expreſſed in whole numbers, as once, twice, thrice, &c; or once, twice and a half, a third, a fourth, or any other fractional number whatever.

E. g. 6 has the ſame Proportion to 24 as 4 has to 16, or 3 to 12, each being contained four times in the other; their Ratio is, therefore, as 1 to 4. 9 has the ſame Proportion to 24 as 6 to 16; for each is contained in the other, twice and two thirds; the Ratio is as 3 to 8. So likewiſe, 27 is to 9 as 21 to 7, or 12 to 4; for, each contains the other thrice; therefore, the Ratio is as 3 to 1. 18 is to 12 as 12 is to 8, for they are, each, as 3 to 2. Thus may the Ratios of any commenſurable Quantities, whatever, be expreſſed by Numbers; and, indeed, ſuch as are incommenſurable, by approximation, are continually approaching nearer to the truth, till the deficiency is leſs than any aſſignable Quantity whatever.

Suppoſe

Suppofe any four Terms or Characters, A, B, C, D, to reprefent four Quantities, that are analogous in their Ratio, two, and two; i. e. when A (any one) has the fame Proportion to B (any other) as C, a third, has to D, the fourth; which is thus written, A : B : : C : D and is read thus; as A is to B, fo is C to D. The four Points, dividing the two Pairs, denotes an equal Ratio between them.

The whole bufinefs, of this fifth Book, is to fhew, what various ways they may be changed, and fo ordered amongft themfelves, that the Proportion arifing, on both fides, may ftill be equal or analogous; which, will admit of great variety; feing, the Ratios of feveral Quantities, that are analogous, may be expreffed by the fame Numbers; as, 2 to 5 expreffes, equally, the Ratio of 6 to 15, of 8 to 20, and of 14 to 35; each being contained in the other, refpectively, twice and one half.

Therefore, either by Alternation, Inverfion, Compofition, Divifion, converting, or mixing the Terms, the Ratio of each, may, in each change, be expreffed by the fame Numbers; & confequently, there is always equality of Proportion. Nor indeed, is it poffible it fhould be otherwife, if the Terms, on both fides, be ordered & changed after the fame manner.

Wherefore, I am and ever was of opinion, notwithftanding what the learned Dr. Barrow, in his mathematical Lectures, and Dr. Keil, in his Preface, fays to the contrary, and the high encomiums which Cunn, in his Preface, beftows on Euclid's Demonftrations of the fifth Book; that, it is involving a thing, in itfelf not very intricate, if not in darknefs and obfcurity, at leaft in perplexity; feeing that, half, or more than half, of the Propofitions may be difpenfed with, or pafs for Axioms; the remainder being fufficient, towards ataining a full Demonftration of the whole.

DEFI-

DEFINITIONS.

Def. I. QUANTITY is whatever may be meaſured or numbered, eſtimated or compared, in reſpect of more or leſs.

As Magnitude or Body; Extenſion, or length ſimply; Weight, or Gravity, Meaſure, Time, Motion, &c.

N. B. One Surface may be equal, in area, to twice, thrice or four times the area of another Surface; or a Line may have twice or thrice, &c. the length of another Line. Alſo, if one Body moves, uniformly, through twice, thrice, or four times the Space, through which another Body moves, in equal Time, it will move with twice, thrice, or four times the velocity. All which, come under the Denomination of Quantity, and may be expreſſed or repreſented by various Symbols or Characters; or by Right Lines, of twice, thrice or four times the length, one of another; or by Numbers; as 6 contains 3, 2, or $1\frac{1}{2}$, twice, thrice, or four times.

Quantities, being compared together, are of two kinds, *viz.* commenſurable, and incommenſurable.

Commenſurable Quantities are ſuch as have a common meaſure; that is, ſuch Quantities as may be divided into the ſame equal Parts, or into Parts of the ſame magnitude.

Two Quantities are commenſurable, when ſome determinate Quantity may be found, which, being taken, or multiplied, certain numbers of times, is equal to either, without deficiency or exceſs.

Incommenſurable Quantities are ſuch as no other Quantity can meaſure; i. e. there cannot be found any determinate Quantity, how ſmall ſoever, which, being multiplied, will be equal to each of the other; but that there will be a deficiency or exceſs, in one or the other.

N. B.

N. B. Any two Quantities, whose Proportion, to each other, can be expressed by Numbers, are commensurable; two Quantities, whose Ratio is such as cannot be expressed in Numbers, are said to be incommensurable to each other.

Def. II. An ALIQUOT, or equal Part, is that, when a less Quantity is contained any number of times, precisely, in a greater; i. e. when a less Quantity, being taken or multiplied any number of times, is equal to a greater.

Thus; an Inch, or a Foot, is an Aliquot part of a Yard or Fathom; or an Ounce of a Pound, &c; 3 is an Aliquot part of 9, 12, or 15, &c.

Def. III. MULTIPLE. That Quantity is called a Multiple, in respect of another Quantity, when it contains, exactly, or is equal to the other, being taken any number of times; then, the less is said to measure the greater.

Thus; a Foot is a Multiple of an Inch, of two, three, four or six Inches, but it is an Aliquot part of a Yard, &c.

A Yard is a Multiple of a Foot or of an Inch, &c. but it is an Aliquot of a Fathom or Furlong, &c.

Def. IV. RATIO, or PROPORTION,* is a mutual habitude or relation of Quantities, of the same kind, in respect to more or less.

Quantities are said to have Ratio to one another, which, being multiplied, can exceed each other.

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* This Definition is rejected, by some, as ungeometrical. Professor Simson says, after a long quotation from Dr. Barrow's 18th Lecture, "that he fully believes, the 3rd and 8th Definitions of the 5th Book (the 4th and 6th of this) are not Euclids, but are added by some unskilful Editor." I must freely own, that I am at a loss to conceive why it is they cavil at them; which, to me, seem both proper and necessary.

Dr. Barrow says, that, "Euclid had, perhaps, no other design in giving this Definition (which he calls metaphysical) than to give a general, though a gross and confused, notion or idea of Ratio to

In order to which, it is neceſſary that they be of the ſame ſpecies or kind, e. g. A Line (having no breath) cannot, by multiplying, be compared with, or be equal to the ſmalleſt ſurface. Alſo a Surface, (having no thickneſs) cannot, by multiplying, be equal to, or compared with a Solid. So neither can Weight be compared with Time, or Motion, &c. Wherefore, ſuch Quantities as are not of the ſame kind, are heterogeneous, and can have no Ratio to one another.

N. B. The Ratio of two Quantities is not expreſſed by the deficiency or exceſs of the Antecedent to the Conſequent; although they may be compared by them, it is not their Ratio.

RATIO, is the Value of the Antecedent Term in reſpect of the Conſequent. (See the next Definition).

Thus, If A is to B as 3 to 5; the Ratio is $\frac{3}{5}$ or 3 fifths of the Conſequent. But, if B be taken for the Antecedent, the Ratio is greater, viz. as 5 to 3; i. e. $\frac{5}{3}$ or 5 thirds, which is one, and two thirds of the Conſequent; whereas, in the former Ratio, the Antecedent, is two fifths deficient of its Conſequent.

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to beginners" grant it; was it not neceſſary, then, to give ſuch a general Idea? which, if it be metaphyſical, cannot, I preſume, with propriety, be called groſs, though it may be confuſed; as, in my Opinion, is every thing relative to Metaphyſics, for, we can only reaſon, on ſuch abſtracted ſubjects, from Analogy, of imaginary Beings with coporeal; which do not, properly, admit of Analogy.

He ſays, further, that nothing in Mathematics depends on it, (the 6th) and that, both might well be ſpared, without any loſs to Geometry." I am aſtoniſhed at ſuch an Aſſertion; does nothing depend on Proportion, or Ratio, and Analogy of Ratios? can we (as he ſays we ſhould principally) attend to thoſe which follow, before we know what is meant by Ratio? can we have any notion of any one, from Alternate Ratio to equality of Ratios, before we know what is meant by Ratio, ſimply? which, in itſelf is undoubtedly metaphyſical; yet, we cannot have any notion of Ratio in abſtract; as the Doctor takes great pains (in his 20th Lecture) to convince us, that Ratios are not Quantities; who ever ſuppoſed it? at the ſame time, I find it impoſſible to form an idea of Ratio, abſtracted from Quantity, which is the Eſſenee of Ratios.

A Ratio

Def. V. ANTECEDENT and CONSEQUENT are the Terms of a Ratio.

In every Ratio there must necessarily be two Terms, denoting two Quantities. The Antecedent is that which is first named, and being referred to, or compared with another Quantity, the latter is called the Consequent.

Thus, A : B, or B : A signifying, that A has some Ratio to B, or B to A, (which it necessarily must if they be supposed to represent two Quantities of the same kind) A or B is the Antecedent, B or A is the Consequent, according, as they are taken, first or last.

A Ratio has no existence, either physically or metaphysically, without Quantity; the Term, Ratio, implies Quantity; where is the Ratio to exist, if not in Quantity? can we conceive a Ratio between non-existences? the supposition is absurd, and cannot be. Therefore, I look on the Definitions of Ratio, and Analogy of Ratios, to be absolutely necessary in Geometry, that they were given by Euclid, and that his Elements of Proportion would be imperfect without them.

Some Geometers make a distinction between Ratio and Proportion; for, Dr. Barrow, in his 4th, and Mr. Stone, in his 8th Definition, say, "Proportion is a similitude of Ratios," which, if Ratio and Proportion be the same, (as the Doctor affirms, in a Note, annexed) is an absurd Definition; viz. Ratio is a similitude of Ratios. Dr. Keil, in his 8th Definition, calls Proportion Analogy, and says, "Analogy is a similitude of Proportions;" and Professor Simson says, "Analogy, or Proportion, is the similitude of Ratios."

Now, in my opinion, the distinction betweem them is so very nice, that it requires very acute discernment to discover the difference. What is by them called Proportion is, properly Analogy, i. e. when the Ratios are equal or similar; for, Dr. Keil, in his 4th Definition, says, "Magnitudes are said to have Proportion to one another." Mr. Stone says, "Magnitudes are said to have a Ratio to one another." Dr. Barrow, in his fifth, changes Magnitudes for Numbers, but makes use of the Term Ratio. So likewise, in the sixth Definition; Keil says, "Magnitudes that have the same Proportion;" Simson and Stone say, in the 6th "Magnitudes which have the same Ratio, are called Proportionals;" consequently, they all mean the same thing, by Ratio and Proportion.

Def. VI. ANALOGY,* of RATIOS, is a similit or equality of Ratios. i. e. When the Antecedent one Ratio contains its Consequent, equally, as the tecedent of another Ratio contains its Consequent.

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* It was not the Design, of those Geometers who made u this Term, to define Analogy, in abstract, but only, to exp what is meant by analogy of Ratios (between two and two Q tities) which is a very proper and expressive Term, for equa of Ratios; it was necessary, that some one or other should be m use of. Analogy, applied to Ratios, is, at least to me, expressive of the thing meant by it; much more so, in my Opini than Proportion; which is only another Term for Ratio, or R for Proportion; which, Terms, are synonimous, and equ expressive. To what purpose are all the various changes, alternation, inversion, &c. if we take away Analogy? all the that is or can be made of them is, only, to shew, that the re of these changes is Analogy; the chief, if not the only wa reasoning from Proportion, in general; consequently it is not unnecessary Term.

If we must needs be caviling, there are other Definitions wh are much more exceptionable; as the fourth of Euclid, "Magnitudes are said to have a Ratio to one another" &c; reality, it is not a Definition of any thing, but is only mean acquaint us, what Quantities can have Ratio to each other; wh common sense has amply provided for: who would ever think comparing Time with Gravity, Measure, &c; or with any c poreal substance, Superficies or Solids?

The fifth is not, properly, a Definition, but intended, only, a sign of Analogy (see the Remark on it, at the end of this Book). The 9th is less so than either of the former, viz. P portion (or Analogous Ratio) consists of three Terms, at le What does this define? nothing; can any Person be at a loss know, that there must needs be three Terms, who has had Analo of Ratios defined, where he always finds four? but, one of four, is sometimes taken twice. I therefore, look on it as sup fluous and unnecessary, seing it is a necessary consequence Analogy; yet, it is not improper, by way of Nota bene or Rema but there could be no impediment, to the attaining of the rest, it was never mentioned. Therefore, it may, very safely, bloted out of the Vocabulary of Definitions; for, to Dr. Barrow's own words, Proportion would sustain no loss by and I do verily believe, that this was never given by Euc as a Definition.

Or, when the Antecedents, of both Ratios, are contained equally in their reſpective Conſequents.

Thus; 9 has the ſame Ratio to 3, as 6 to 2; for each is as 3 to 1, i. e. each Antecedent contains its Conſequent thrice; therefore, there is the ſame Ratio or Proportion between them, and they are ſaid to be Analogous.

Alſo, 6 is to 9 as 2 to 3, for each Antecedent is contained once and a half in its Conſequent.

When the ſecond Term has not the ſame Proportion to the third, as the firſt has to the ſecond, or the third to the fourth, they are ſaid to be directly or diſcretely proportional.

Thus, 4:6::8:12, directly or diſcretely; but, 4.6::9:13½ is both directly and continual. (See Def. 7.)

When one Antecedent contains its Conſequent, or is contained in it, more than the other Antecedent contains or is contained in its Conſequent, though by ever ſo ſmall a part, the Ratio is unequal; and they are called, ſimply, UNEQUAL RATIOS.

Analogy of Ratios is thus expreſſed; A:B::C:D.

The firſt and third Terms, A and C, are the Antecedents, and may be taken for any Quantities whatever; B and D, the third and fourth, are the Conſequents.

Wherefore, if A be ſuppoſed, in Numbers, to repreſent 9, and C, 3; and, if B be equal 6, it will be thus, 9.6::3:2; conſequently, D will repreſent 2. But, if B alſo repreſents 3, then will D expreſs or ſignify but 1, viz. 9:3::3:1.

Again; if B repreſent a greater Quantity than A, D will alſo be greater than C; thus, 9:12::3:4; or 9:15::3:5.

The three firſt Terms being fixed or known Quantities, and in a certain order, the fourth is a neceſſary conſequence.

N. B. In Analogy of Ratios, there muſt neceſſarily be three Terms, ſeing that there is ſimilitude or equality of Ratios, (and each Ratio conſiſts of two Terms, the Antecedent and Con-

ſequent; Def. V.) But, ſince the Conſequent of the firſt Ra
may, alſo, be the Antecedent of the ſecond (as A:B::B:
three Terms are ſufficient to conſtitute Analogy; in which ca
it is neceſſary that they are all of the ſame ſpecies or kind.

But, when there are four Terms (A;B::C:D) the fir
Ratio (A:B) may be of a different ſpecies from the other (C:D
for, there is the ſame Ratio between a Foot and an Inch, two
three, or four Inches, &c. as between a Pound, and one, two
three, or four Ounces, (Troy Weight) conſequently, there ma
be Analogy of Ratios, between heterogeneous Quantities.

Def. VII. CONTINUAL RATIO is when, in thre or more Quantities, the firſt is to the ſecond, as the ſecon to the third, as the third to the fourth, &c.; or, mor properly, the third is to the fourth, as the ſecond to th third, and that, as the firſt is to the ſecond.

Thus. In four or more Quantities, A B, C, D, if has the ſame Proportion to D, as B has to C; and, if has alſo the ſame Proportion to C, as A has to B, they a then in continual Proportion, or geometrical Progreſſio and are thus writ, or ſymbolized; A:B:C:D ∺.

N. B. Each Term being twice taken, except the firſt and laſt, ar therefore, both Antecedent and Conſequent, i. e. the ſeco Term is a Conſequent in reſpect of the firſt, and the third the ſecond; but they are, alſo, Antecedents in reſpect of t third and fourth Terms.

In continual Ratio, all the Terms are, neceſſarily, of t ſame ſpecies or kind. i.e. The Quantities muſt be homogene

If A and C, the two firſt Terms, or D and C, the two laſt, any fixed or determinate Quantities, all the reſt are de minable; which, in Numbers, is obvious.

Let A be 4, and B 6; it will then be thus, 4:6:9:13½ ∺ & each conſequent Term containing its Antecedent once a half. But if A be to 3, and B 6, it will be 3:6:12:24 the Ratio being as 2 to 1, or double.

Or, if A repreſents 2, and B 6, then 2:6:18:54 ∺ &c. Ratio being triple, or as to 3 to 1. So that, whatever R there is between the firſt and ſecond, that Ratio is continu

There F

Therefore, when the Ratio is increasing (or rather when the Quantity is increasing, for the Ratio is the same, throughout) the Quantity of C, D, &c. is determined either by Addition or Multiplication; as in a decreasing Ratio, (i. e. when the Antecedent is the greater Quantity) by Division or Subtraction.

Def. VIII. PROPORTIONALS are such Quantities as are in the same Ratio; discretely or continual.

Proportionals consist, at least, of three Terms, or Quantities; which are usually symbolized by Right Lines; or by Characters, as A:B::C:D, the Ratio being determined by Numbers.

Def. IX. MEAN PROPORTIONAL, and THIRD PROPORTIONAL.

When there are three Terms, only, or, if three Terms are Proportionals, as A, B, C, they are in continual Proportion; thus, if A is to B, as B to C, in equal Ratio; then, the middle Term, B, is a Mean, between the other two; and is, therefore, called a MEAN PROPORTIONAL.

The first and last Terms are the Extremes of the Proportionals; either of which, A or C, when there are but three, is called, a THIRD PROPORTIONAL, in respect of the other two.

Hence it is evident, that, in continual Proportion, of four or more Quantities, each Term, except the first and the last, is a Mean, between its Antecedent and Consequent.

N. B. If two Terms or Quantities are given, and a third be required, in the same Ratio to either of the given Terms as that has to the other, the Quantity found is a third Proportional.

Def. X. EXTREME and MEAN PROPORTION is, when any Quantity is so divided, into two Parts, that, the lesser, the greater, and the whole, are in continual Ratio.

A Right

A Right Line is ſaid to be divided in extreme and mea
Proportion; when, being cut into two unequal Parts
the Ratio of the whole Line, to the greater Segment, is th
ſame as of the greater Segment to the leſs.

If the Line AB be cut in C. A C

Then, if the whole Line, AB, is to AC as AC is to CB
AB is divided in extreme and mean Proportion.

Def. XI. A FOURTH PROPORTIONAL.

If four Terms, A, B, C, and D, are Proportionals
whether they are analogous, only, or in continual Pro-
portion, either of the Extremes, A or D, is a fourth
Proportional, in reſpect of the other three.

N. B. If three Terms or Quantities are given, and a fourth is required, in the ſame Ratio, to any one of the given Quantities, as the remaining two have, the one to the other; the Quantity, ſo found, is a fourth Proportional.

Def. XII. DUPLICATE and TRIPLICATE RATIO.

If four, or more Terms, A, B, C, D, are in continual Proportion; the firſt, A, is ſaid to have, to the third, C, a duplicate Ratio of A to B, the firſt to the ſecond.

And the Proportion of A to D, the firſt to the fourth, is triplicate of A to B; and ſo on.

Def. XIII. ALTERNATE RATIO is the mutation of the two middle Terms, of four proportional Quantities;* i. e. when Antecedent is compared with Antecedent, and Conſequent with Conſequent; and they are called homo-logus, corational, or correſponding Quantities.

Thus, A:B::C:D, directly. — If 27:18::6:4.
Then, A:C::B:D, alternately — So, 27:6::18:4.

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* Alternate Ratio cannot exiſt or take place, unleſs both Ratios are of the ſame ſpecies; ſeeing, there is no Ratio or compariſon between

N. B. When the Product, arising from the multiplication of the two middle Terms, is equal to that of the two Extremes, it is a certain proof of Analogy, consequently, it will be the same, whether they are taken directly or alternately.

Def. XIV. INVERSE RATIO is when the Consequent is taken for the Antecedent, and compared with the Antecedent as a Consequent.

In this Case, the middle Terms become the extremes.

If A:B::C:D directly. — 27:18::6:4.

Then, B:A::D:C inversely. — So, 18:27::4:6.

Def. XV. COMPOUNDED RATIO is when the Antecedent and Consequent, being taken or added together, as one Quantity, are compared either with the Antecedent or with the Consequent.

If A:B::C:D; then, A+B:A, or B,::C+D:C, or D.

6:9::2:3; then, 15:6, or 9, :: 5 : 2, or 3.

Def. XVI. DIVIDED RATIO is when the difference, between the Antecedent and the Consequent, is compared either with the Antecedent or the Consequent.

If A:B::C:D. Then, whether A be greater than B, or B than A; the difference, being taken, is compared either with A or B; as also, the difference between C and D, is compared with C or D.

Thus, A—B, or B—A,: A, or B,::C—D, or D—C:, C or D,

15:6::5:2; then, the difference, 9,:6, or 15,::3:2, or 5.

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between Quantities that are heterogeneous. (see Def. IV.). For, if four Quantities are in analogous Ratio, (A:B::C:D,) it is not necessary that the first pair (A, B,) should be of the same kind as the last (C and D.) But, in Alternate Ratio, the Antecedent of one Ratio is compared with the Antecedent of the other, and Consequent with Consequent; wherefore, it is absolutely necessary that both Ratios are of the same kind.

N. B. In the seven following Definitions, it is the same whether they are of the same kind or not; i. e. each pair may be of a different kind and heterogeneous to the other.

Def. XVII.

Def. XVII. CONVERSE RATIO is when the Antecedent is compared with the difference between the Antecedent and the Conſequent.

If A:B::C:D. Then, whether A exceeds B, or B exceeds A: it will be thus; A:A—B, or B—A, ::C:C—D, or D—C. 12:27::4:9; and, 12 or 27, :15 (the difference)::4, or 9:5.

Def. XVIII. MIXED RATIO is when the Antecedent added to the Conſequent, as one Quantity, is compared with the exceſs, by which the one exceeds the other;

And, vice verſa, when the difference, between them, is compared with the ſum of both.

If A:B::C:D; then, A+B:A—B, or B—A, ::C+D :C—D, or D—C. 7.3::21:9; then, 10:4::30:12, the Antecedent being the greater Quantity.

Def. XIX. ORDINATE RATIO of EQUALITY.

If there be three or more Quantities in one Rank or Order, in any Ratio whatever, and as many in another Rank, in the ſame Ratio, comparing two and two; i. e. as the firſt is to the ſecond, in one Rank, ſo is the firſt to the ſecond, in the other; and, as the ſecond is to the third, ſo is the ſecond to the third, &c.; it will then be as the firſt is to the laſt, in one Rank, ſo is the firſt to the laſt in the other; or (comparing the Extremes) as the firſt is to the firſt, ſo is the laſt to the laſt.

In the two Ranks A, B, C, D, and E, F, G, H;
If, A:B::E:F, and B:C::F:G; alſo, as C:D::G:H.
Then, as A:C, or D, ::E:G or H; alſo, as A:E::C:G, or D:H.

Let the firſt Rank be as 3:7:5:8; and let E be 9
Then, F will be 21, G will be 15, and H 24.
For, as 3:7::9:21; and as 7:5::21:15; alſo, as 5:8::15:24
Then 3:5, or 8, as 9:15, or 24; and as 3:9::5:15, or 8:24

Rati

Ratio of equality * is of two kinds, ordinate and inordinate.

Ordinate Proportion is what is already defined; or, when there are, at firſt, but four Quantities, in analogous Ratio, either diſcretely or continual; and other Quantities are aded, after the ſame manner, and in equal Ratio.

For example; let the Ratio of C:D be as A:B.

Now, if other Quantities be added, on both ſides, in ſuch wiſe, that the Ratio is the ſame, ordinately, there is ſtill equality.

Firſt, let the Quantities added be Antecedents;
If A:B::C:D; then, let E be to A as F to C;
or, let them both be Conſequents, viz. as B:G::D:H.
It is manifeſt, that, in either Caſe, the Ratio is ſtill the ſame; being taken progreſſively.

For, as E:B, or A:G,::F:D, or C to H;
alſo, as A:B, or C:D,::E:F, or G to H; in either.

Def. XX. INORDINATE RATIO of EQUALITY is, when in equal Ratios of two Quantities each; as the Conſequent of one Order is to ſome other Quantity, ſo is another Quantity to the Antecedent of the other Order.
Then, as the Antecedent, of the firſt, is to that other Quantity, ſo is the other Quantity to the Conſequent of the other Order.

If A:B::C:D; and, if B:E::F:C; then, A:E::F:D.
3:4::21:28; and, 4:14::6:21; then, 3:14::6:28.
Again, if E:G::H:F; then, A:G::H:D.
For, 14:7::12:6; then, 3:7::12:28.

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* By Ratio of equality is not meant equality of Ratios ſimply; but, in two Ranks of Quantities, whoſe Ratios are ordinate or inordinate, the Ratios are equal, at equal Diſtances from the Extremes.

Def. XXI. Composition of Ratios.*

In any Number of Quantities, and in any Order, whatever; the Ratio of the first to the last is equal to that which is compounded of all the intermediate Ratios.

If A, B, C, and D, represent as many Quantities whatever is the Ratio of A to B, of B to C, and of C to D; the Ratio of A to D, is equal to the Ratio compounded of them all.

Thus, if A is to B as 3 to 5, B to C as 5 to 8, and C to D as 8 to 6.

Then, $A : D :: 3 : 6$; compounded of $\frac{A}{B}\frac{3}{5}$ of $\frac{B}{C}\frac{5}{8}$ & of $\frac{C}{D}\frac{8}{6}$

i.e. Multiply all the Antecedents into one another; $3 \times 5 \times 8 = 120$; and divide the Sum, by that of all the Consequents; $5 \times 8 \times 6 = 240$, then is A to D, as 120 to 240, or as 3 to 6; i.e. as 1 to 2, or $\frac{5}{10}$.

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* This last Definition is generally given in the sixth Book; for which I cannot conceive a reason. The Doctrine of Proportion is a distinct subject, and therefore, all which relates to it should be together, in one Book; which, according to Euclid, is the fifth: to which (in the use and application of it, in the sixth and other Books,) we may refer on all Occasions.

Dr. Barrow, indeed, gives nearly the same thing, in both. In Def. 20. B. 5. he says, "any number of Magnitudes being put; the proportion, of the first to the last, is compounded out of the proportion of the first to the second, the second to the third, and the third to the fourth, &c. till the proportion arise."

The proportion, which is between the first and the last, will and must necessarily, arise, by the composition of all the intermediate Ratios, which is evident from the manner of compounding them; nor is it possible to be otherwise; seeing that, each of the intermediate Quantities is both Antecedent and Consequent. Therefore, the Ratio emerging at last, seeing, it depends entirely on the first and last, must ever be equal to the Ratio of the first to the last; let the intermediate Quantities be ever so many, and in what Proportion soever.

According

According to Euclid, there are six various ways of reasoning, from Proportion, including ordinate and inordinate Ratio in that of Equality; which, in effect, is all one. Inordinate Ratio differs from ordinate, in nothing but the disposal of the last Quantity; for, whether it be considered and compared as an Antecedent or as a Consequent, by placing it first or last; the Ratio, in either case, being the same, it will still be as the first to the last, so is the first to the last; but not, one Quantity, (added) to the other, as in the former case, where the Quantities, added, are both Antecedents, or both Consequents.

But there is another way of reasoning by Proportion, distinct from the rest, which is not in Euclid; viz. mixed Ratio, (Def. XVIII.) that is, comparing the sum of the Antecedent and Consequent with their difference. In respect of Converse Ratio, it differs in nothing from inverse, which includes all Converse Ratios whatever; since, all Ratios, that are analogous, are inversely so, or conversely, which is the same thing. Nor do I see any Reason, why the 15th Definition, of Compounded Ratio, has not a converse, as well as the 16th of Divided Ratio; seeing that, one holds good as well as the other; and is as necessary to be known, it being as frequently made use of as the other.

That the manner, and the order, of the several changes may be seen at one view, I have given them over again; with the converse of each, in the following abstract; keeping the same Ratio throughout the whole. In which five Cases, viz. alternating, inverting, compounding, dividing and mixing, with their converse, together with equality, ordinate and inordinate, is contained all the variety in which the Terms, of analogous Ratios, may be changed and still be analogous; and which, answers almost every purpose, of reasoning from Proportion.

If A:B::C:D, directly, as 3 to 2; thus, 21:14:: 9:
Then, A:C::B:D, alternately. - - - 21: 9::14:
Alſo, B:A::D:C, inverſely. - - - - 14:21:: 6=

And, A+B:A, or B,::C+D:C, or D.
35 : 21, or 14, :: 15 : 9, or 6. } by Compoſiti

Alſo; A, or B, : A+B ::C, or D,: C+D.
21, or 14, : 35 :: 9, or 6,: 15 } converſely.

And, A—B : A, or B,::C—D :C, or D.
7:21, or 14,:: 3 : 9, or 6, } by Diviſion.

Alſo; A, or B, : A—B::C, or D:C—D
21, or 14: 7 :: 9, or 6 : 3 } converſely.

Again, A+B:A—B::C+D:C—D, mixing; 35: 7::15:
or, A—B:A+B::C—D:C+D, converſely, 7:35:: 3:1

Ordinate Ratio of equality.

If A : B::C:D; and, if B:E ::D: F; then, A: E::C: F
21:14::9: 6; and, 14:42:: 6:18; then, 21:42:: 9:18

Inordinate Ratio of equality.

If A : B::C:D; and, if B: E::F:C; then, A: E::F:D
21:14:: 9: 6; and, 14:42:: 3: 9; then, 21:42:: 3: 6

Thus, having ſhewn all the variety of changes, and, tha the Ratios emerging from each, in numbers, are equal c analogous; there remains, one might reaſonable ſuppoſ nothing more to be done. For, after this knowledge onc obtained, of what uſe can it be to perplex the Student, wi a tedious Demonſtration of what, in Numbers, is clear ar manifeſt; and which, I am perſuaded, needs no furth Demonſtration. Nor do I ſee, excepting Alternate Rati (which is, alſo, ſufficiently evident, in Numbers) that a of the reſt require Demonſtration; or are made more ma feſt by it. For, the Ratios of all commenſurable Quantit may be expreſſed by Numbers; and it is ſufficiently manife that, by Analagy, it muſt alſo hold true in reſpect of Qua tities, which are incommenſurable.

It is evident, that, since A:B::C:D, whether the Ratio of A to B be commensurable or incommensurable, the Ratio of C to D being the same, as A to B, they will still be analogous in every change.

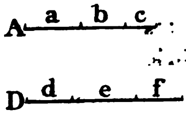
E. g. Suppose the Ratio of A to B and of C to D be as the Side of a Square to the Diagonal; or, as the Diameter of a Circle is to its Circumference, &c.

Now, it is manifest, that the Side of one Square is to its Diagonal, as the Side of any other Square is to its Diagonal; consequently, their Ratios are equal, notwithstanding they cannot be expressed in Numbers; for 3 : 5 :: 9 : 15, i. e. 3 : 5 :: 3 : 5; consequently, 3 : 3 :: 5 : 5, &c. Therefore, since the Side of one Square, is to its Diagonal, as the Side of any other Square to its Diagonal; consequently, as Side is to Side, so is Diagonal to Diagonal; also, as the Side added to the Diagonal, is to either Side or Diagonal; so is any other Side added to its Diagonal, to either; likewise as one Diagonal is to the Side, so is the other Diagonal to the Side; or, as the Diagonal, less the Side, to either, so is the Diagonal, less the Side, to either; and thus it must ever be through all the variety of Changes that can be, if they are ordered and changed after the same manner, on both Sides.

In what I have advanced, on the Subject of Proportion, is contained the essence of the whole fifth Book of Euclid; and if it be clearly understood, the Doctrine of Proportion is already acquired; and consequently, all that can be said more of it, by way of Demonstration, is, in a great measure, superfluous. However, not to leave the whole Doctrine of Proportion without a Foundation, least it should be censured by the scrupulous, I have demonstrated or more fully illustrated, some of the most essential, on which the whole depends; some of the others I have made Axioms, for being self evident they require no Demonstration.

AXIOMS,

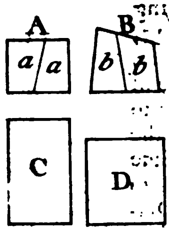
AXIOMS, or self-evident PROPOSITIONS.



Ax. I. One Quantity is to any other Quantity, as all th Parts of one, is to all the Parts of the other.

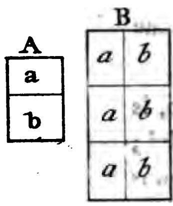
Let A and D be two Quantities, and, let A divided, any how, into Parts, a, b, c; also let be divided any how, in d, e, and f.

Then, A : B :: a + b + c : d + e + f.



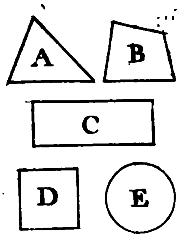
II. Equimultiples,* or equal Parts, of equal Quantiti are equal.

Let A and B be equal Quantities; then, if and D be equal twice or thrice, &c. of A and B respectively; 2 A = 2 B, or 3 A = 3 B, i. e. C = D Also, if A and B be divided into two or mor equal Parts, *a a*, *b b*, &c; *a* = *b* or ½ A = ½ B, &c.



III. Any Multiple of the whole is equal to the sam Multiple of all its Parts, taken together.

Let B be taken any Multiple of A; then, what ever Multiple B is of A, it is manifest, that contains the same Multiple of a and b, the Par of A and B; a, *a*, *a*, and b, *b*, *b*.



IV. The seventh Proposition of Euclid.

Equal Quantities have the same Ratio to a third Quantity, or to equal Quantities.

For, if A and B be equal Quantities, they ha the same Ratio to a third Quantity, C, or to eq Quantities, D and E; seeing that, either may taken for the other.

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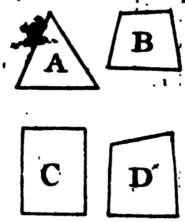
* By Equimultiples is meant, that the Quantities are ta or multiplied, an equal number of times.

V.

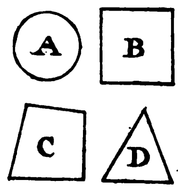
V. The ninth Propoſition of Euclid.



Quantities that have an equal Ratio to the ſame Quantity, or to equal Quantities, are equal.

If A and B have the ſame Ratio to C, or D; A and B are equal. Alſo, if A has the ſame Ratio to C, as B to D; if C=D, A=B.

VI. The tenth Propoſition of Euclid.

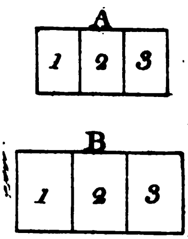


That Quantity which has the greater Ratio to a third Quantity, or to equal Quantities, is the greater Quantity. And, of two Quantities, that, to which a third Quantity has a greater Ratio, is the leſſer Quantity.

If C has a greater Ratio to B, than A has to B, C is greater than A.

Alſo, if A has a greater Ratio to D than to C, D is leſs than C.

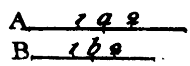
VII. The fifteenth Propoſition of Euclid.



Quantities have an equal Ratio to their Equimultiples; or to their equal Parts.

Let A and B be two Quantities, in any Ratio whatever; and let them he taken an equal number of times. Or, let them be divided into the ſame number of equal Parts; The wholes of A and B are equimultiples of thoſe Parts,

$A : 2A$ or $3A$, $::$ $B : 2B$ or $3B$. i.e. $1 : 2$ or $3 :: 1 : 2$ or 3.

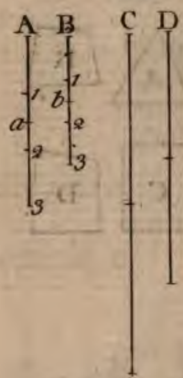


Or, as $\frac{1}{2}A$, or $\frac{1}{3}A$, $: A ::$ $\frac{1}{2}B$, or $\frac{1}{3}B$, $: B$.

Alſo, $A : Aa\ (\frac{1}{2}A)$ or $A1$, $::$ $B : Bb\ (\frac{1}{2}B)$ or $B1$.

VIII. Quantities are in the ſame Ratio, to each other, as their Equimultiples; or, as their equal Parts.

Let

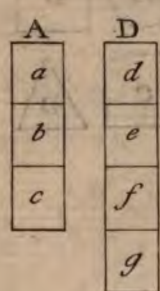


Let A and B be any Quantities, divided into equal Parts, as before, at *a* and *b*, or 1, 2, 3; alſo, let C and D be any equimultiples of A and B.

Then, A : B :: 2A : 2B, or as 3A : 3B; i. e. as C : D.

For C and D are any equimultiples of A and B; wherefore, as often as A contains B, or is contained in B; ſo often 2 A contains 2 B, or is contained in 2 B; i. e. as C contains D, or is contained in D; conſequently, A : B :: C : D.

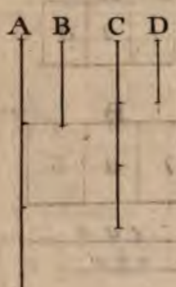
Alſo, as A : B :: A*a* ($\frac{1}{2}$A) : B*b* ($\frac{1}{2}$B)
or, as A1 to B1. ($\frac{1}{3}$ A to $\frac{1}{3}$ B.)



IX. If two Quantities be divided into Parts of equal magnitude (conſequently equal amongſt themſelves) then, one Quantity is in the ſame Ratio to the other, as the number of Parts, in one, to the number of Parts, in the other.

Let A and D be two Quantities, divided into equal Parts, *a*, *b*, and *c*; *d*, *e*, *f*, and *g*;

Then, A : D :: 3 : 4; or, as 6 to 8, &c.
i. e. as $a + b + c +$ is to $d + e + f + g$.

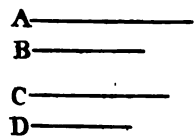


X. If Quantities are proportional; the firſt to the ſecond, as a third is to a fourth, &c; then, if the firſt be any multiple or equal part of the ſecond, the third is an equimultiple or equal part of the fourth, &c.

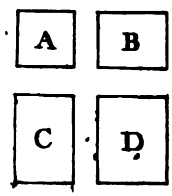
For Quantities are to each other, as the number of Parts in one, to the number of Parts in the other. - - - - by the 9th.

Therefore, if A:B::C:D; A contains, or is contained in B, as often as C contains, or is contained in D. - - - - - Def. 6.
otherwiſe, the Ratios are not equal, or analogous.

Axiom XI. In four proportional Quantities, i. e. when any one is to another, as a third is to the fourth, (Def. 6.) whether the firft be equal to, greater, or lefs than the fecond, the third is alfo equal to, greater, or lefs than the fourth.



If A:B::C:D; then, if A be equal to B, C=D; and, if A be greater, or lefs than B; C is, alfo, greater, or lefs than D.

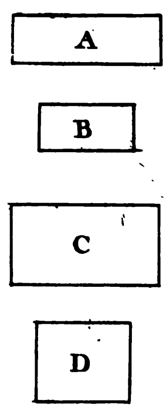


For, A has the fame Ratio to B, as C to D;

And, a greater Quantity cannot have the fame Ratio to a lefs, as a lefs to a greater. - Def. 6.

XII. The fourteenth Propofition of Euclid.

If four Quantities are proportional; then, if the Antecedent of one Ratio be greater than the Antecedent of the other, the Confequent of the firft is alfo greater than the Confequent of the other; if the firft are equal, the other are alfo equal; and, if lefs, lefs.



For, A:B::C:D; and, if A be equal to, greater, or lefs, than B, C is alfo equal to, greater, or lefs than D. - - - - - - 12

Wherefore, if A=C, B=D (Ax. 5.) and confequently, if A be greater or lefs than B, C is, neceffarily, greater or lefs than D.

XIII. The eleventh Propofition, of Euclid.

Ratios that are equal to the fame Ratio, or to qual Ratios, are equal between themfelves.

This follows from the third Axiom of Book 1ft, fubftituting Ratios for Quantities.

XIV. The thirteenth Propoſition of Euclid.

If four Quantities are proportional, the firſt to the ſecond as the third to the fourth; and if the Ratio of the third to the fourth, be greater or leſs than a fifth Quantity has to a ſixth; the Ratio of the firſt to the ſecond, is alſo greater or leſs than the fifth to the ſixth.

If A:B::C:D; and, if the Ratio of C to D be either greater or leſs than E to F; the Ratio of A to B is alſo greater or leſs, than E to F.

For, the Ratio of A to B is equal to the Ratio of C to D therefore, their Ratios are the ſame, in reſpect to any other.

Note. By miſtake, the 7th and 8th Axioms are miſplaced. But as either is the other, alternately, it is of no conſequence; ſave only, that the firſt part of the 8th is, properly, Euclid's fifteenth Propoſition.

POSTULATES.

1. Grant, that equal Ratios may be taken, one for the other.
2. Grant, that any Quantity may be divided into any number of Parts, equal to one another; or as any other Quantity is divided.
3. Grant, that a Quantity may be taken or aſſumed, in any Ratio to any given Quantity.

The aſſumption of this laſt Axiom is not allowed by ſome Geometers; notwithſtanding, to me, it appears full as poſſible, that equimultiples, or equal parts of Quantities may be take by one or other of which Suppoſitions, they demonſtrate the whole fifth Book.

THE

THEOREM I. 1 Euclid.

If Quantities, which are equimultiples or equal parts of other Quantities, be added into one Sum, or Quantity; the ſame multiple or part, which one Quantity is of the other, reſpectively, the whole is of the whole.

Let A, B, and C be equimultiples, or equal parts, of D, E, and F, reſpectively; i. e. whatever multiple, or part, A is of D, let B be the ſame multiple, or part, of E, and C of F.

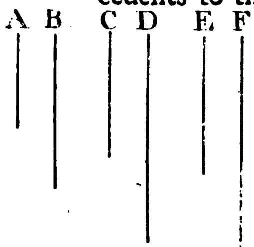
Then, A, B, and C, added together, into one Sum, is the ſame multiple or part of D, E, and F, added together, as A is of D, B of E, or C of F.

DEM. For, let A be equal to $2D$, or $3D$; then, $B = 2E$, or $3E$, &c. conſ. $A+B+C = D+E+F$, taken twice, or thrice, &c. But, Quantities are to each other, as the number of Parts, in one, is to the number of equal Parts, in the other; -Ax. 9. wh. as $A:D$, $B:E$, and $C:F$, $::A+B+C:D+E+F$. Conſequently, as often as A contains D, or is contained in D, &c. ſo often the whole of $A+B+C$ contains, or is contained in, $D+E+F$. Therefore, &c. Q. E. D.

For, if either be multiples of the other; the latter is, conſequently, equal parts of the former.

THEOREM II. 12 Euclid.

If any number of Quantities, are Proportionals, (i. e. being taken two and two, their Ratios are analogous) then, as any one of the Antecedents is to its Conſequent, ſo is the ſum of all the Antecedents to the ſum of all the Conſequents.



Let A, B, C, D, E, and F be proportional Quantities; of which, let A be to B, as C is to D, and as E to F.

I ſay, as A is to B, or C to D; ſo is A, C, and E, added together, to B, D, and F, together.

DEM. Now, $A:B::C:D$, and, as $E:F$, therefore, their Ratios are equal, - - - - - Def. 6.
Wherefore, if $A=B$, $C=D$, and $E=F$ - - Ax. 11.
conſequently, $A+C+E=B+D+F$. - - Ax. 6. 1.
therefore, as $A:B$, or as $C:D$, $::A+C+E:B+D+F$.

Again, if A be any multiple, or equal part, of B; then, C is an equimultiple, or equal part, of D, & E of F -Ax. 10.
wherefore, $A+C+E$ is the ſame multiple, or equal part, of $B+D+E$, as A is of B, as C of D, and as E of F -Th. 1.

But, Quantities have the ſame Ratio, to each other, as their Equimultiples, or as their equal Parts. - - Ax. 8.
Th. as $A:B$, as $C:D$, or as $E:F$, $::A+C+E:B+D+F$.

Now, becauſe there are ſuch Quantities, which cannot be equimultiples or equal Parts, one of the other, this Demonſtration is not poſitive, in reſpect of ſuch Quantities; and, notwithſtanding they may be divided into the ſame number of equal Parts, yet it will be found no eaſy matter, to give perfect Demonſtration, ſeeing that, equal parts are in the ſame Ratio to each other as the wholes. Ax. 8th.

Suppoſe A to B, and C to D, &c. are incommenſurable, but in the ſame Ratio; ſtill, $A+C+E:B+D+F$, &c. $::A:B$, &c. becauſe, A might either be equal to C, or E, or it might be ſome Multiple or equal Part of C, and C of E, &c.

Or, ſuppoſe A be incommenſurable to C, and C to E, &c. provided A and B, C and D, &c. be commenſurable, the ſame thing may be done; as in the Theorem.

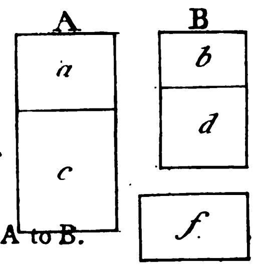
But, being incommenſurable both ways, there cannot poſſibly be equality in either; and the Demonſtration given (or rather preſumed) from a greater or leſs Ratio, is by no means poſitive; ſeeing, it is not aſcertained in what Degree they are ſo.

For, A may have to B, and C to D, &c. the Ratio of the Side of a Square to its Diagonal, or as the Diagonal to the Side, &c. and A to C may be the ſame; or, as the Diameter of a Circle to the Circumference; and C to E, in extreme and mean Proportion.

Yet it is eaſy to conceive, from what has been proved, that (let the Ratio of A to B be what it may) ſince the Ratio of C to D, and of E to F, &c. is the ſame, conſequently, $A+C:B+D$, or $A+C+E:B+D+F::A:B$, or C to D, &c. and although it is not poſſible to give direct proof, in ſuch Caſe, yet we may ſafely conclude that it is ſo; and, on that preſumption, all which follows will be found poſitive, let the Ratio be what it may.

THEOREM III. 19 Euclid.

In two Quantities, if a Part taken from one, be in the ſame Ratio to a Part taken from the other, as one Quantity is to the other; the remainders of thoſe Quantities are alſo in the ſame Ratio, as one Quantity is to the other.



Let A and B be two Quantities, in any Ratio whatever, and, ſuppoſe any part of A be taken (as a) from A.

Let b be taken, from B, in the ſame proportion to a, as A to B.

I ſay, the remaining Part c, is to d, as A to B.

Let any other Quantity (f) be ſo taken, $a:b::c:f$. Poſt. 3.

DEM. Now, $a:b::c:f$; and $a:b::A:B$; conſ. $c:f::A:B$. - Ax. 13

But, $A:B::a+c:b+d$ (Ax. 1.) and, as $a:b:c:f$. - Poſt.

wherefore, $a+c:b+f::a:b$; i. e. as $A:B$. - Th. 2

conſ. $a+c:b+f::a+c:b+d$; wh. $b+f=b+d$. - Ax. 4

And by taking away b, which is common, $f=d$. - Ax. 7. 1

But, $c:f::a:b$ (Poſt.) conſequently, $c:d::a:b$. - Ax. 4

And, $a:b::A:B$ (Hyp.) Therefore, $c:d::A:B$. - Ax. 13

COR. If one Quantity be to another as a third is to a fourth; and, if a fifth be to the ſixth, as the third to the fourth; then, the firſt and fifth, added together, has the ſame Ratio to the ſecond and ſixth, together, as the third has to the fourth.

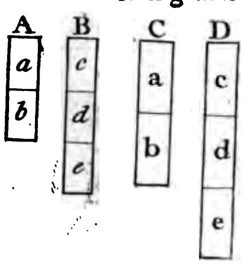
THEOREM IV. 16 Euclid.

If four Quantities, of the ſame kind, are Proportionals; i. e. taken two and two, in a certain order, they are directly proportional, or analogous in their Ratio; they are alſo analogous when taken alternately.

If A is to B, as C is to D; then, A is to C, as B to D.

DEM. Now, ſince A : B : : C : D; whether A be equal to, greater or leſs than B; C is alſo equal to, greater or leſs than D. - - - - - - - Ax. 11.
And, if A be equal to, greater, or leſs than C;
B is alſo equal to, greater, or leſs than D. - - - 12.
But, A has the ſame Ratio to B as C to D; - - Hyp.
it follows, then, that A has the ſame to C, as B to D.

For, whether A and B be equimultiples, or equal parts of C and D; A is in the ſame Ratio to C, as B to D-Ax. 7. ſeeing that A is to B as C to D. - - - - - 8.



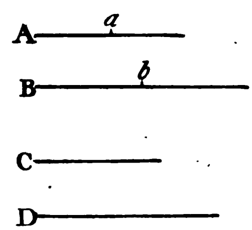
Suppoſe the Ratio of A to B be as 2 to 3; and, imagine them divided into equal Parts, in *a* and *b*; *c*, *d*, and *e*.

Alſo, ſuppoſe C and D divided into the ſame number of equal Parts, reſpectively, in a and b; c, d, and e.

Then

Then, ſince A:B::2:3, & A:B::C:D; C:D::2:3 - Ax. 13
wherefore, the Parts, a and b, (of C) and c, d, e, (of D) are all equal amongſt themſelves. - - Ax. 9.
Conſequently, as *a*:a, or *b*:b, ::*c*:c, *d*:d, or *e*:e. - Ax. 4.
But, *a* and *b* are equal parts of A; and, a and b, of C.
Alſo, *c*, *d*, & *e*. are equal parts of B; and, c, d, and e, of D.
But, Quantities are, to each other, in the ſame Ratio as their equimultiples, or equal parts. - - - Ax. 8.
Therefore, as A is to C, ſo is B to D. Q. E. D.

Again. Suppoſe the Ratio of A to B, and of C to D, be incommenſurable; and conſequently, cannot be divided into Parts, which are all equal amongſt themſelves;



Then, let A and B be divided into the ſame number of equal Parts, ſuppoſe they are biſected, in *a* and *b*. - - Poſt. 2.

Now, A:B::C:D, and A:B::*a*:*b*, - - - Ax. 8.
conſequently, C:D::*a*:*b*, - - - - Ax. 13.
But, A:*a*::B:*b* (Ax. 7) and, as *a*:*b*::C:D. - as above.
Wherefore, by ſubſtituting one for the other, - Poſt. 1.
conſequently, as A is to C, ſo is B to D. Q. E. D.

THEOREM V.

Quantities, which are analogous when taken directly, are alſo analogous, taken inverſely.

If A is to B, as C is to D; then, B is to A, as D to C.

EM. For, if A:B::C:D; then, A:C::B:D, - - by 4.
i. e. B:D::A:C; then, B:A::D:C. - - ſame.

THE-

THEOREM VI. 18 Euclid

Quantities which are analogous, taken directly, are also analogous, when compounded.

If A is to B as C is to D.
Then A added to B, is to A or B; as C added to D, is to C or D.

DEM. For, since A:B::C:D; then, A:C::B:D - Th.
wh. as A:C::A+B:C+D. (2) conf. A:A+B::C:C+D
and consequently, as A+B:A::C+D:C. - - Th.
But, B:D::A:C; therefore, A+B:B::C+D:D - Ax.

COR. By inversion, A, or B, :A+B::C, or D, :C+D.

THEOREM VII. 17 Eucl.

Quantities which are analogous, directly, are also analogous when divided.

If, in four Quantities, A, B, C, and D; A is to B, as C to D
Then, according as A or B is the greater Quantity,
A less B (or B less A) is to A, or B,
as C less D (or D less C) is to C, or D.

DEM. Now, A:B::C:D. (Hyp.) wherefore, A:C::B:D - 4
But, as A:C, (or B to D)::A—B (or B—A):C—D - 3.
conf. A (or B) :A—B::C (or D) :C—D - - - 4.
and, conf. A—B :A (or B) ::C—D:C (or D) - - 5.
Or; - - B—A:A, or B, ::D—C:C, or D.

COR. The Converse of this is first proved, in the third step.

THE-

THEOREM VIII.

If four Quantities are Proportionals, in a certain order, they are alſo proportional when mixed; i.e. their Sums and Differences are analogous.

If A is to B as C to D.

Then, $A+B:A-B$ (or $B-A$) $::C+D:C-D$ (or $D-C$)

DEM. Now, $A:B::C:D$ (Hyp.) then $A:C::B:D$ - Th. 4.
Wherefore, as $A:C::A+B:C+D$ - - - - 2.
alſo, as $A:C::A-B:C-D$. - - - - 3.
conſequently, $A+B:C+D::A-B:C-D$ - - Ax. 13.
therefore, $A+B:A-B::C+D:C-D$. - Th. 4.

COR. $A-B:A+B::C-D:C+D$, by inverſion. - 5.

THEOREM IX. 22 Euclid.

If there be three or more Quantities, and others equal to them in number; which, taken two and two in a certain order (ordinately, or progreſſively) are in the ſame Ratio; then will the firſt be to the laſt, as the firſt to the laſt, in each Order of Quantities. Alſo, as the firſt is to the firſt, ſo is the laſt to the laſt.

If $A:B::C:D$; and, if $B:E::D:F$; A is to E as C to F.

DEM. Now, $A:B::C:D$; wherefore, $A:C::B:D$ ⎫
And - $B:E::D:F$; wherefore, $B:D::E:F$ ⎭ - Th. 4.
Now, ſince $A:C::B:D$; and, as $B:D::E:F$.
conſ. $A:C::E:F$ (Ax. 13.) Therefore, $A:E::C:F$. - 4.

N. B. The laſt Part is proved firſt.

Mm COR. 1.

Again. If other Quantities, G and H, be added, in the same Ratio to the last Consequents, respectively; as E is to G, so is F to H; it is manifest, that A is to G, as C to H; the first, still, in the same Ratio to the last.

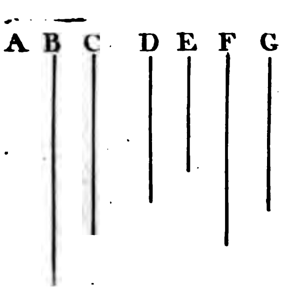
For, as A:E::C:F; and, as E:G::F:H; by Theorem. wherefore, A:C::E:F; and, as E:F::G:H; by the 4th. Conf. as A:C::G:H (Ax. 13) th. A:G::C:H - same.

And thus it will ever be, if Quantities be added (in equal Ratio) after the same Order, ad infinitum.

Cor. Hence, if the Quantities, added are in equal Ratio to the Antecedents, respectively;
thus, if A:B::C:D, and, if E be to A as F to C;
it is manifest, that as E is to B, so is F to D, &c.

THEOREM X. 23 Euclid.

In two Ranks or order of Quantities, if there be three or more in each; which, taken two and two, inordinately, are in the same Ratio; then, as the first is to the last, in one Order, so is the first to the last in the other; as by equality.



Let A, B, C, and D, E, F, be Quantities in such Order, so that, as A is to B, so is E to F; and, as B is to C, so is D to E.

I say, that, as A is to C, so is D to F.

Let any other Quantity, G, be taken so; that, as D is to E, so is F to G; - Post. 3.
conf. as D is to F, so is E to G. - - 4.

DEM.

DEM. Then, as before, A:B::E:F, wh. A:E::B:F. - 4.
Alſo, as B:C::D:E, i.e. as F:G, wh B:F::C:G-ſame.
Now, A:E::B:F; and B:F::C:G; wh. A:E:C:G-Ax.13.
and conſequently, as A:C::E:G. - - - - Th. 4.
But, D:F::E:G, (above) Th. as A:C::D:F - Ax.13.

If other Quantities be added, inordinately, in equal Ratio, it will ſtill be as the firſt is to the laſt, in one Rank, ſo is the firſt to the laſt, in the other. e. g.

Let the Quantities, G and H, be ſo taken, that, as C is to G, ſo is H to D; then, as A is G, ſo is H to F.

For, as A:C::D:F (above) and, as C:G::H:D - Hyp. conſequently, as A is to G, ſo is H to F; - - as before.

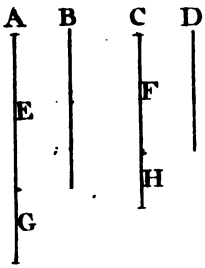
Inordinate equality may be proved, (after ordinate) by means of the 21ſt Definition, viz. by compoſition of Ratios; but, as that Definition is ſomewhat obſcure, the Demonſtration ariſing, therefrom, would be ſo too.

For, ſince A:B::E:F; and, as B:C::D:E; alſo as D:G::H:D; it is manifeſt, ſeeing that there is equal Ratios between the two extremes, A and G, H and F; let the intermediate Ratios be ever ſo many, and in what order ſoever, provided there be an equal number in each Rank; and the extreme Ratios are equal, ordinately or inordinately; alſo, that there be found equal Ratios between the Extremes, in each Rank; two and two, any how ſituated; the Ratio, compounded of them all, is, as the firſt to the laſt, in each Rank, or order of Quantities; by the Definition.

But, the extreme Ratios are equal (ordinately or inordinately) and conſequently, the extreme Quantities, of both Ranks, are in equal Ratio.

THEOREM XI. 25 Euclid.

If four Quantities are Proportionals, in a certain order; the greatest and the least are, together, greater than the other two.



Let A, B, C, and D, be four Quantities, of which, let A be the greatest; and let them be, as A to B, so is C to D; consequently, D, is the least Quantity.

For, if the first be greater than the second, the third is greater than the fourth - Ax. 1. and, if the first be greater than the third, the second is greater than the fourth. - 2.

Dem. Because A and C are greater Quantities than B & D; let E and F be taken from A and B, respectively, equal to B and D, respectively.

Now, A:B::C:D. But E=B, and F=D - Con.
wherefore, as A:E::C:F. - - - Ax. 4.
conf. A—E (i. e. G) :A::C—F (i. e. H) :C - Th. 7.
i. e. G:A::H:C; wherefore, G : H :: A : C - Th. 4.
But, A is greater than C; wh. G is greater than H - Ax. 1.
conf. G+E (i. e. A) +D, is greater than H+F (i.e. C) + B.
Th. A+D is greater than C+B, the two middle Terms.

In the last ten Theorems, and the Corollaries deducible from them, is contained all the various ways of reasoning, by Proportion, which I conceive to be useful; two of which, the fifth and eighth, are not proved by Euclid. The eighth, indeed, is not mentioned, by him, or any of his Followers, that I am acquainted with; but, I am somewhat surprized, that none

none have given a Demonſtration of Inverſe Ratio; which being proved, every Converſe Ratio is alſo proved. Profeſſor Simſon is aware of that deficiency, and has very judiciouſly introduced it, as an additional Propoſition (B;) after which, I muſt think it needleſs, to add the Converſe of divided Ratio (E) unleſs he had, alſo, given the Converſe of all.

But what, to me, ſeems very extraordinary is, that Euclid ſhould not demonſtrate either; alſo, that he ſhould prove (if it may be ſaid to be proved) divided Ratio (the 17th) before compounded (the 18th) when the latter is, in ſome meaſure, made a Condition of the former; they are, indeed, Converſe, each of the other. In the 17th he ſays, "If Magnitudes, taken jointly or compounded, be proportional, they are alſo proportional taken ſeparately, or divided." Now, by the tenor of theſe Premiſes, *if* one is, the other is, alſo; and conſequently, *if* one is not, the other is not; whereas, the *Condition* is preſumed, only; and conſequently, there is no proof of either. But where is the neceſſity of proving it on that Condition, when it may be done as elegantly, ſolidly, and briefly without it, from the ſimple Analogy of the two Ratios.

Either I am not clear, in the nature and meaning of compounded and divided Ratio, or thoſe Propoſitions are not to the purpoſe.

By Definitions 15th and 16th is to be underſtood, when Quantities are proportional (ſimply) they are alſo proportional compounded or divided; which, Profeſſor Simſon has expreſsly ſaid, in his Definitions of them; and which, I have demonſtrated in the 6th and 7th Theorems. Now, his 17th Propoſition ſays, as above; and *vice verſa* in the 18th, viz. that, if Quantities are Proportionals, when compounded or divided, they are ſo, divided or compounded; which, I cannot conceive to be the true meaning, but as I have ſtated them, in the Theorems, and have demonſtrated, on that Hypotheſis.

If what I have advanced, on the ſublime Doctrine of Proportion, be not ſufficient, for any purpoſe whatever, I will be bold to ſay, that it is not ſo in Euclid, or any of his Commentators. The Axioms, which I have given, are gathered from the moſt eaſy and ſimple ideas of Proportion; after the Definitions are digeſted and clearly underſtood, there is no Perſon would heſitate, one moment, to grant every Axiom; they require no proof, being ſelf evident.

That Quantities are in the ſame Ratio as their Equimultiples, or equal Parts (the 8th) is manifeſt to the meaneſt capacity; and this is one of the principal, on which Euclid has founded his whole Theory of Proportion; which amounts to no more than this; that one is to two, three, &c. as, two, is to four, ſix, &c; or, as three to ſix, nine, &c. i. e. as one, to two, three, &c. Or, that one Quantity is to any other Quantity, as the half, or third part, &c. of one, is to the half, or third part, &c. of the other.

The 1ſt Axiom, in the firſt Book of Elements, of Euclid, (the 3rd of theſe Elements) expreſly ſays "*Things*, which are equal to the ſame *Thing*, are equal between themſelves." Now, the queſtion is, whether *Ratios* are *Things?* Certainly, if any thing be meant by Ratios, they muſt come within that Appellation; for, granting the Term to be, in itſelf, metaphyſical; yet, *ſomething* is underſtood and meant by Ratio, and conſequently, *Ratios* are *Things*. Then, the 11th Propoſition of Euclid's fifth Book, is as much an Axiom as the firſt of the firſt; and, I have accordingly made it ſo, in the thirteenth. The remainder of the Axioms are as ſimple, obvious, and ſelf evident; which, with the Poſtulates, being granted, the whole of the Theorems, I am confident, will be found ſolidly, yet briefly, and clearly demonſtrated.

A critical

A critical Remark on Euclid's fifth Definition, of the fifth Book.

Euclid in his fifth Definition, has given us a Criterion, whereby to determine Analogy of Ratios, rather than a ſimple Definition of what it is.

He ſays, "Magnitudes (by which is meant Quantities of any kind) are ſaid to be in the ſame Proportion, the firſt to the ſecond, and the third to the fourth, when the equimultiples of the firſt and third, being compared with the equimultiples of the ſecond and fourth, according to any multiplication; either together exceed the one the other, or together are equal the one to the other, or together are leſs, the one than the other." This is, litterally, according to the old Tranſlation by H. Billinſley in 1570.

Dr. Barrow ſays, "Magnitudes are in the ſame Ratio, when the Equimultiples of the firſt and third, compared with the equimultiples of the ſecond and fourth, according to any multiplication whatſoever, either both together are leſs than the ſecond and fourth both together, or equal taken together, or exceed one the other together, if thoſe be taken which anſwer one to the other."

Dr. Keil, from the latin tranſlation of Commandine, has nearly the ſame words, viz. "the Equimultiples of the firſt and third (compared as above) are either both together greater, equal, or leſs than the equimultiples of the ſecond and fourth; if thoſe be taken that anſwer each other."

Cunn did well to explain the meaning of Dr. Keil's Definition, which is literally this; that, equimultiples of the firſt and third, taken or added together (i. e. into one Sum or Quantity) are either equal, greater, or leſs than the equimultiples of the ſecond and fourth. I think, there is great occaſion for a comment on Dr. Barrow's; which, to me, is quite unintelligible.

After all that has been ſaid, concerning this Definition, by the learned Doctor and others, either the original Text muſt be very obſcure, or the expounders have rendered it ſo, for I declare, I cannot think the words bear ſenſe. But, giving the greateſt latitude poſſible to the meaning of the words, they imply, the firſt and the third taken together, and compared with the ſecond and fourth, together, (i. e. in one Sum) in which Caſe it amounts to a clear Axiom; for they muſt, certainly, be either equal, greater, or leſs, one than the other; after which, the words, "if thoſe be taken which anſwer the one to the other," mean nothing at all.

Mr.

Mr. Stone ſays, "they are, together, either deficient, equal or exceed *each* other." They may, indeed, be equal, *each* to the other; but, it is impoſſible that they can, at the ſame time, be either greater or leſs, *each* than the other.

The word, *together*, ſeems, to me, quite ſuperfluous; or elſe the tenor of it is, in this place, *at the ſame time*, or, *by the ſame multiplication*; which is quite foreign to the common acceptation of it. It is, perhaps, better to omit it entirely, and ſupply its place with *both* (which Barrow and Keil have added) it is then plain; the firſt and the third, compared with the ſecond and fourth, will be either *both* greater, *both* equal or *both* leſs; which is the concluſion all have drawn from it; and then, the words, if thoſe be taken which anſwer each other, ſeem neceſſary.

Mr. Stone in vindicating and illuſtrating this Definition, ſhews, that it is clearly deduced from the fourth and fourteenth Propoſitions. According to the Idea I have of the Definitions of Terms, in any Science, they ought to be defined by their moſt natural and ſimple properties; from which the demonſtrations of ſome Theorems are to be obtained; for it ſeems inconſiſtent, firſt to demonſtrate ſeveral preceding Propoſitions, by means of a certain Definition, previous to the demonſtration of others, from which the Definition, itſelf, is to be deduced, and demonſtrated.

It is true, he ſays "it is not ſo ſimple and plain as the Definition of Numbers, or that which might be given of commenſurable Magnitudes." And again: "Euclid could not have given any other, ſo elegant and general a Definition, that would have taken in incommenſurable Magnitudes, as well as Numbers, and commenſurable ones.

It is well known, that the Diagonal of a Square is not commenſurable to its Side; alſo, the Circumference of a Circle is incommenſurable by its Diameter or Radius; that is, their Ratios cannot be expreſſed in Numbers; for, the Side of a Square cannot be divided into any number of parts, though ever ſo ſmall, which will be an aliquot part of the Diagonal, i. e. they cannot both be divided into the ſame equal parts, or into parts of equal magnitude, without defect; ſuch Quantities are, therefore, incommenſurable.

The Ratio, between the Side of a Square, and its Diagonal is very different from that, between the Diameter of a Circle and its Circumference. Yet, the Ratio between the Side of every Square and its Diagonal, or between the Diameter of every Circle and its Circumference, is the ſame; i. e. their Ratios are equal or analogous.

It is intimated above, and is ſtrongly ſupported by Dr. Barrow and others, that this Definition comprehends or extends to Incommenſurables

commensurables. I am, however, persuaded, that it is no easy matter to determine the analogy of such Ratios by it. The rule it prescribes, or is intended to prescribe, is simply this. Take any Equimultiples of the first and third Terms, in four Quantities, that are presumed to be analogous in their Ratios; i. e. multiply the two Antecedent Terms, equally, the same number of Times, and, either by the same Number or any other, multiply the two Consequents, the third and fourth, or take Equimultiples of them; then, the first and the third, being compared with the second and fourth; if the first exceeds the second, the third will also exceed the fourth, if one be equal, the other is also equal, or they will be both less, than the second and fourth; i. e. the two Antecedent Terms will, at the same time, be either both equal, both greater, or both less than their respective Consequents; and, when this is the case, in all multiplications, *whatever*, we may then conclude that their Ratios are analogous.

Now, if the Ratios are of incommensurable Quantities, which cannot be expressed by Numbers, those Quantities cannot be multiplied by Numbers; and, in Lines, it would require greater accuracy than it is possible for any person to apply, to take Equimultiples of them. It is certain, that being multiplied, mentally, according to this Definition, if they were analogous, the Antecedents would be either greater or less than the Consequents, continually, but never equal; for if, by taking Equimultiples, they could be equal, they may, consequently, also be divided into equal Parts; but, it is well known, they cannot; which, negative Quality, is the distinguishing characteristic of Incommensurables. And, since it is manifest that the Antecedents of Ratios which are not analogous may, by multiplying, be either greater or less than their respective Consequents, but never both equal, at the same time, i. e. by the same multiplication; how, then, is it possible to discover analogy of Ratios by this Criterion? seeing that, in Ratios of commensurable Quantities, which are not analogous, they may be greater or less, but never equal, as they will ever be in incommensurable Quantities that are analogous.

Hence, it appears, to me, that, to produce equality, of the Antecedent to the Consequent, is the only certain and infallible sign of Proportionality, or analogous Ratio, between commensurable Quantities; but, since that Criterion fails, in Incommensurables, there cannot, by its means, be any positive determination concerning Analogy, in such Quantities; unless, by trying *all multiplications*, *whatever*, they are, *at all times*, either *both* greater, or *both* less.

If the Antecedents are either both greater, equal, or less than their respective Consequents (and no Person would look for Proportionality otherwise) it is manifest, that, by multiplying the

If what I have advanced, on the ſublime Doctrine of Pro-portion, be not ſufficient, for any purpoſe whatever, I will be bold to ſay, that it is not ſo in Euclid, or any of his Commentators. The Axioms, which I have given, are gathered from the moſt eaſy and ſimple ideas of Proportion; after the Definitions are digeſted and clearly underſtood, there is no Perſon would heſitate, one moment, to grant every Axiom; they require no proof, being ſelf evident.

That Quantities are in the ſame Ratio as their Equimultiples, or equal Parts (the 8th) is manifeſt to the meaneſt capacity; and this is one of the principal, on which Euclid has founded his whole Theory of Proportion; which amounts to no more than this; that one is to two, three, &c. as, two, is to four, ſix, &c; or, as three to ſix, nine, &c. i. e. as one, to two, three, &c. Or, that one Quantity is to any other Quantity, as the half, or third part, &c. of one, is to the half, or third part, &c. of the other.

The 1ſt Axiom, in the firſt Book of Elements, of Euclid, (the 3rd of theſe Elements) expreſly ſays "*Things*, which are equal to the ſame *Thing*, are equal between themſelves." Now, the queſtion is, whether *Ratios* are *Things?* Certainly, if any thing be meant by Ratios, they muſt come within that Appellation; for, granting the Term to be, in itſelf, metaphyſical; yet, *ſomething* is underſtood and meant by Ratio, and conſequently, *Ratios* are *Things*. Then, the 11th Propoſition of Euclid's fifth Book, is as much an Axiom as the firſt of the firſt; and, I have accordingly made it ſo, in the thirteenth. The remainder of the Axioms are as ſimple, obvious, and ſelf evident; which, with the Poſtulates, being granted, the whole of the Theorems, I am confident, will be found ſolidly, yet briefly, and clearly demonſtrated.

A critical

A critical Remark on Euclid's fifth Definition, of the fifth Book.

Euclid in his fifth Definition, has given us a Criterion, whereby to determine Analogy of Ratios, rather than a simple Definition of what it is.

He says, "Magnitudes (by which is meant Quantities of any kind) are said to be in the same Proportion, the first to the second, and the third to the fourth, when the equimultiples of the first and third, being compared with the equimultiples of the second and fourth, according to any multiplication; either together exceed the one the other, or together are equal the one to the other, or together are less, the one than the other." This is, litterally, according to the old Translation by H. Billinsley in 1570.

Dr. Barrow says, "Magnitudes are in the same Ratio, when the Equimultiples of the first and third, compared with the equimultiples of the second and fourth, according to any multiplication whatsoever, either both together are less than the second and fourth both together, or equal taken together, or exceed one the other together, if those be taken which answer one to the other."

Dr. Keil, from the latin translation of Commandine, has nearly the same words, viz. "the Equimultiples of the first and third (compared as above) are either both together greater, equal, or less than the equimultiples of the second and fourth; if those be taken that answer each other."

Cunn did well to explain the meaning of Dr. Keil's Definition, which is literally this; that, equimultiples of the first and third, taken or added together (i. e. into one Sum or Quantity) are either equal, greater, or less than the equimultiples of the second and fourth. I think, there is great occasion for a comment on Dr. Barrow's; which, to me, is quite unintelligible.

After all that has been said, concerning this Definition, by the learned Doctor and others, either the original Text must be very obscure, or the expounders have rendered it so, for I declare, I cannot think the words bear sense. But, giving the greatest latitude possible to the meaning of the words, they imply, the first and the third taken together, and compared with the second and fourth, together, (i. e. in one Sum) in which Case it amounts to a clear Axiom; for they must, certainly, be either equal, greater, or less, one than the other; after which, the words, "if those be taken which answer the one to the other," mean nothing at all.

Mr.

Mr. Stone ſays, "they are, together, either deficient, equal or exceed *each* other." They may, indeed, be equal, *each* to the other; but, it is impoſſible that they can, at the ſame time, be either greater or leſs, *each* than the other.

The word, *together*, ſeems, to me, quite ſuperfluous; or elſe the tenor of it is, in this place, *at the ſame time*, or, *by the ſame multiplication*; which is quite foreign to the common acceptation of it. It is, perhaps, better to omit it entirely, and ſupply its place with *both* (which Barrow and Keil have added) it is then plain; the firſt and the third, compared with the ſecond and fourth, will be either *both* greater, *both* equal or *both* leſs; which is the concluſion all have drawn from it; and then, the words, if thoſe be taken which anſwer each other, ſeem neceſſary.

Mr. Stone in vindicating and illuſtrating this Definition, ſhews, that it is clearly deduced from the fourth and fourteenth Propoſitions. According to the Idea I have of the Definitions of Terms, in any Science, they ought to be defined by their moſt natural and ſimple properties; from which the demonſtrations of ſome Theorems are to be obtained; for it ſeems inconſiſtent, firſt to demonſtrate ſeveral preceding Propoſitions, by means of a certain Definition, previous to the demonſtration of others, from which the Definition, itſelf, is to be deduced, and demonſtrated.

It is true, he ſays "it is not ſo ſimple and plain as the Definition of Numbers, or that which might be given of commenſurable Magnitudes." And again: "Euclid could not have given any other, ſo elegant and general a Definition, that would have taken in incommenſurable Magnitudes, as well as Numbers, and commenſurable ones.

It is well known, that the Diagonal of a Square is not commenſurable to its Side; alſo, the Circumference of a Circle is incommenſurable by its Diameter or Radius; that is, their Ratios cannot be expreſſed in Numbers; for, the Side of a Square cannot be divided into any number of parts, though ever ſo ſmall, which will be an aliquot part of the Diagonal, i. e. they cannot both be divided into the ſame equal parts, or into parts of equal magnitude, without defect; ſuch Quantities are, therefore, incommenſurable.

The Ratio, between the Side of a Square, and its Diagonal is very different from that, between the Diameter of a Circle and its Circumference. Yet, the Ratio between the Side of every Square and its Diagonal, or between the Diameter of every Circle and its Circumference, is the ſame; i. e. their Ratios are equal or analogous.

It is intimated above, and is ſtrongly ſupported by Dr. Barrow and others, that this Definition comprehends or extends to Incommenſurables

commensurables. I am, however, persuaded, that it is no easy matter to determine the analogy of such Ratios by it. The rule it prescribes, or is intended to prescribe, is simply this. Take any Equimultiples of the first and third Terms, in four Quantities, that are presumed to be analogous in their Ratios; i. e. multiply the two Antecedent Terms, equally, the same number of Times, and, either by the same Number or any other, multiply the two Consequents, the third and fourth, or take Equimultiples of them; then, the first and the third, being compared with the second and fourth; if the first exceeds the second, the third will also exceed the fourth, if one be equal, the other is also equal, or they will be both less, than the second and fourth; i. e. the two Antecedent Terms will, at the same time, be either both equal, both greater, or both less than their respective Consequents; and, when this is the case, in all multiplications, *whatever*, we may then conclude that their Ratios are analogous.

Now, if the Ratios are of incommensurable Quantities, which cannot be expressed by Numbers, those Quantities cannot be multiplied by Numbers; and, in Lines, it would require greater accuracy than it is possible for any person to apply, to take Equimultiples of them. It is certain, that being multiplied, mentally, according to this Definition, if they were analogous, the Antecedents would be either greater or less than the Consequents, continually, but never equal; for if, by taking Equimultiples, they could be equal, they may, consequently, also be divided into equal Parts; but, it is well known, they cannot; which, negative Quality, is the distinguishing characteristic of Incommensurables. And, since it is manifest that the Antecedents of Ratios which are not analogous may, by multiplying, be either greater or less than their respective Consequents, but never both equal, at the same time, i. e. by the same multiplication; how, then, is it possible to discover analogy of Ratios by this Criterion? seeing that, in Ratios of commensurable Quantities, which are not analogous, they may be greater or less, but never equal, as they will ever be in incommensurable Quantities that are analogous.

Hence, it appears, to me, that, to produce equality, of the Antecedent to the Consequent, is the only certain and infallible sign of Proportionality, or analogous Ratio, between commensurable Quantities; but, since that Criterion fails, in Incommensurables, there cannot, by its means, be any positive determination concerning Analogy, in such Quantities; unless, by trying *all multiplications*, *whatever*, they are, *at all times*, either *both* greater, or *both* less.

If the Antecedents are either both greater, equal, or less than their respective Consequents (and no Person would look for Proportionality otherwise) it is manifest, that, by multiplying the

Antecedents any number of times, equally, and the Consequents the same, or any other, also equally; they must necessarily be, at the same time, either both greater or both less; but, if they are not analogous, they never can, at the same time, be both equal, i. e. each Antecedent to its respective Consequent; and therefore, no other Sign, but equality, is necessary, or sufficient, to prove Analogy of Ratios.

I am persuaded, that either Euclid, himself, or his Advocates are under some mistake in respect of this fifth Definition; which does not, positively, determine the Analogy of incommensurable Quantities. Therefore, the investigation, by means of it, is vague and imperfect; and consequently, all that is built, entirely on it, is so too; seeing it cannot, as it is presumed, discover Analogy, in incommensurable Quantities; because, equality of the Antecedent to the Consequent (the only certain Sign) cannot, by multiplying, be produced.

It is manifest, that if Equimultiples of incommensurable Quantities are taken, they will still be in the same Ratio; i. e. if the side of a Square and its Diagonal, &c. be equally multiplied; but no multiples of the Side and of the Diagonal can be so taken, that they may be equal to one another; for, it is evident, if they could, they might, also (as observed above) be divided into the same equal Parts, which cannot be. And, for such as are commensurable, it is an unnecessary and unsatisfactory method of determining.

Then, since no sign, but equality of the Antecedent to the Consequent, can be positive, and since that cannot subsist between Incommensurables, although it is certain, that *if* one is, the other must necessarily be so too; but since that, *if*, can never possibly happen, we cannot, on that supposition, determine Analogy of Ratios, in such Quantities.

The Reverend Dr. Barrow, in his 21st Lecture, has made very learned Defence of this Definition; and, spiritedly, encounters the Detractors of Euclid, viz. Ramus, Tacquet, and Hobbs, in the 22nd. But, having silenced those weak Opponents, he is close put to it by Borellus, in his 23rd and last Lecture; where, after various skirmishes, in which, the Doctor has not always the advantage, he briefly sums up the whole Evidence, in these words, "that, in his judgment, there is nothing extant in the whole work of the Elements, more subtilly invented, more solidly established, or more accurately handled, than the Doctrine of Proportionalities." Nevertheless, if his own Definition of it is to be the Touchstone, I cannot subscribe to his Attestation.

Now, as I have before observed, in the Preamble to this fifth Book, Proportion is, in a great measure, an innate principle; which being clearly explained, what is meant by it, and explicated by Numbers

bers, is ſo very intelligible, that, to any tolerable Capacity, it does not require other Demonſtration; and all that is built on that knowledge is, I am confident, as ſolid and permanent, and as ſecurely eſtabliſhed, as by all the Demonſtration which can poſſibly be given; at leaſt, on the foundation of Euclid's fifth Definition.

In reſpect of commenſurable Quantities, to all which Numbers may be applied, whether their Ratios are analogous or not, is readily diſcovered; ſince, in Analogy, the Ratios being equal, the ſame Numbers will expreſs either.

E.g. 56:98::12:21; which being reduced to their loweſt Denomination, the Ratio, of each, is determined. The firſt pair, 56 and 98, being divided, ſeparately, by 14, gives 4 and 7, the true Ratio of that pair. Now, if the other pair produces the ſame, they are analogous; 12 and 21, divided ſingly, by 3, produce, alſo, 4 and 7; which is, therefore, the true Ratio of both; conſequently they are analogous, ſeeing the Ratios are equal.

If there be taken Equimultiples of Ratios that are analogous, i.e. being multiplied various ways, either all by one Number, (any number whatever) or the firſt pair by one Number, and the ſecond pair by another; or, if the firſt and third be multiplied by one Number, and the ſecond and fourth by any other, the Products ariſing, from every ſuch multiplication, are ſtill analogous, in the Ratio of both Pairs.

Let A : B :: C : D, in the Ratio of 3 to 5.
Thus, 9 : 15 :: 3 : 5.

So, 27 : 45 :: 9 : 15, being all multiplied by 3.
Alſo, 18 : 30 :: 15 : 25; the firſt Pair by 2, and the ſecond by 5.

In both theſe Caſes, the Ratio is ſtill as A to B, or as C to D.

Again, 36:30::12:10, $\left\{\right.$ the firſt and third, being multiplied by 4, are both greater than the ſecond and fourth, multiplied by 2,

Alſo, 45:45::15:15, $\left\{\right.$ the firſt and third multiplied by 5, the ſecond and fourth by 3; in which caſe they are both equal.

And, being all multiplied by any one Number, as above, it is evident they will both be leſs. Or, being multiplyed by various Numbers, they will both be leſs; which illuſtrates the fifth Definition of Euclid.

Now, it is very obvious, in Numbers, from any one, of theſe various operations, that there is analogy of Ratios between A and B, C and D; and it is full as obvious without. But, provided it is not known, whether the Ratios are analogous or not, no one, nor all of the three firſt trials are ſufficient to determine it, merely from the firſt and third being, at the ſame time, both greater or both leſs than the ſecond and fourth; becauſe, thoſe things may, and very frequently do, happen, where there is not Analogy.

E. g. Let the Ratio of A to B be as 3 to 4; and, C:D::2¾:4. In whole Numbers, A:B::3:4, or 12:16, and C:D::11:16.

Now, let there be taken Equimultiples of A and C, and other equimultiples of B and D. e. g. Let the Antecedents be taken thrice, and the Consequents twice; it will be 3 A : 2 B :: 9 : 8, and 3 C : 2 D :: 33 : 32; by which multiplication, it is evident that the Antecedents are both greater than their Consequents.

But, seeing that the Ratios are not equal, it is impossible to take Equimultiples of the two Antecedents, and also Equimultiples of the Consequents, by which, the Products of the Antecedents shall be both equal, respectively, to their Consequents; and also, because their Ratios are not equal, such Equimultiples may be taken of them, that one Antecedent shall be greater, and the other less, than its respective Consequent.

E. g. Let A and C be taken four times, and B and D thrice, it will then be, 4A:3B::12:12, and 4C:3D::44:48;

Again; let A and C be taken 7 times, and B and D 5 times; it will be 7A : 5B :: 21 : 20; and, 7C : 5D :: 77 : 80.

Either of these operations is sufficient to evince, that the Ratios are not analogous; i. e. A is not to B, as C to D; for, it is impossible that equality of Antecedent to Consequent should happen in both pairs, (i. e. of A to B, and of C to D), unless they are analogous; which is the only indisputable proof of Analogy; for, one Quantity must necessarily be to itself, as any other Quantity is to itself. And since it is not possible for equality to happen, by multiplication any more than by division, between incommensurable Quantities, it is manifest, that Analogy of such Ratios cannot be absolutely determined by that Criterion.

Wherefore, the Demonstration, according to Euclid, in the fifth Book is not positive, in respect either of commensurable or incommensurable Quantities; but is only presumptive, at best, and is very obscure and unsatisfactory; since it is manifest, that, in unequal Ratios, Equimultiples may be so taken (according to Euclid) in which the Antecedents will be either both greater or both less than their Consequents (as it has been shewn above) but, it is not possible they can be taken, so, that both Antecedents shall be equal to their Consequents.

I am therefore of Opinion, that, the Doctrine of Proportion is not properly investigated by Equimultiples; nor can I conceive, that taking multiples of Quantities, in order to prove the equality of their Ratios, is properly geometrical. It seems, to me, more consistent to compare Quantities by themselves, either whole and entire, or by their equal Parts.

But it is manifest, that there are Ratios, which will not admit of division into the same Parts, i. e. into Parts of the same magnitude; yet it is certain, that they may, with equal ease and more propriety, be divided into the same number of Parts, as to take Equimultiples of them; which Method I have therefore adopted, as the most rational and eligible.

ELEMENTS OF GEOMETRY.

BOOK VI.

THE ſixth Book of Elements is not only the moſt uſeful and valuable of the whole, but alſo the moſt entertaining and inſtructive. In it, the Doctrine of Proportion is applied to real uſe, in finding out the more extraordinary properties of Plane Figures, (particularly Triangles and Parallelograms) by the moſt ſubtle and ſolid reaſoning that can be conceived: by means of which, many excellent and extenſively uſeful Problems are found out, and applied to various and moſt notable purpoſes.

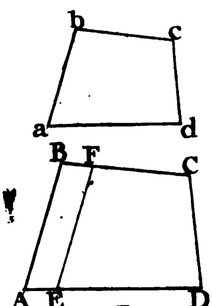
In it, the Rule of Three, or Proportion, called by way of eminence, the Golden Rule, has its foundation and exiſtence (Theo. 9.) in it, is alſo demonſtrated, with more facility, that, not only the Square of the Hypothenuſe is equal to the two Squares of the Sides, of every right angled Triangle, but likewiſe, that it is ſo, in all ſimilar Figures whatever, regular or irregular, conſtructed on the three ſides, for the correſponding ones of each Figure. (Theo. 16.) In it, is determined the juſt proportion which exiſts between two ſimilar Figures of any kind, and exhibited by two Right Lines (12 and 13) by which means, any Figure, whatever, may be conſtructed, in any Ratio to a given one, (Prob. 37 and 16;) or, the Side of a Square may be determined, whoſe Area ſhall be equal to that of any given right lined Figure, whatever. (Prob. 24 and 25.) And, what is more extraordinary, a right lined Figure may be conſtructed ſimilar to any given Figure, and equal to any other, of different Forms and Denominations, being right lined. (Prob. 28.)

 In

In it, ſeveral valuable properties of the Circle are briefl
demonſtrated; in ſhort, to ſee and admire all which it con
tains, I recommend the careful and attentive peruſal of it
as the readieſt and moſt certain means, by which its beauti
and excellencies can poſſibly be communicated.

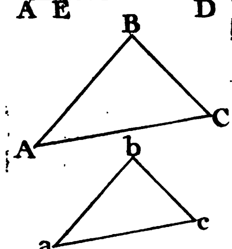
DEFINITIONS.

Def. I. SIMILAR FIGURES are ſuch as hav all their Angles equal, each to the other, reſpectivel and alſo, the Sides which are oppoſite to equal Angle or which lie between equal Angles are proportional.



The Quadrilateral a b c d, having all i Angles equal, reſpectively, to the Angles c the Quadrilateral ABCD, viz. a equal A b equal B, &c; and, if, the Side a b hav the ſame Ratio to AB, as b c has to BC, &c. then, a b c d is ſimilar, or like, ABCD.

If EF be drawn parallel to A B, th Angles at E and F, are equal to A and B reſpectively (4. 1.) conſequently to a and b wherefore, all the Angles of the Quadrila tateral a b c d, are equal to thoſe of EFCD but, ſeeing the correſponding Sides, a and E F, bc and FC, &c. are not pro portional with c d and CD, a b c d is no ſimilar to E F C D.



Triangles ABC, a b c, havingall the Angles of one, equal reſpectively, to the Angles of the other, have their Sides neceſſarily proportional, and are, conſequently, ſimilar Figures.

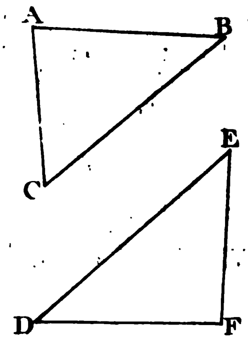
The Sides a b, AB; b c, BC, &c. being oppoſite to equal Angles, or which lie between equal Angles, are HOMOLOGOUS or correſponding Sides.

N. B. All ordinate Figures, whatever, i. e. ſuch as are equilatera and equiangular, one with another, are ſimilar Figures; as equi lateral Triangles, Squares, &c. Alſo all Circles are ſimilar Figures

Def. II. In Triangles, and Parallelograms, if there be two Sides, in one, which, with two Sides in another, of the fame kind, and about the fame Angle, are Proportionals; in fuch wife, that the Antecedent of one, and the Confequent of the other pair, are in the fame Figure, thofe Figures are faid to be Reciprocal.

But more properly, the Ratio is reciprocal in fuch Figures; for, reciprocal Figures is unmeaning.

In the Triangles ABC, DEF; if the Sides AB and BC of the one, and DE, DF, of the other, are Proportionals; and, if AB is to DE, as DF, of the fame Triangle, is to BC of the other; then, the Proportionals are reciprocal in thofe Figures; and they are ufually, though improperly, called reciprocal Figures.



Underftand the fame of Parallelograms.

N. B. Equiangular, or fimilar, Triangles are not reciprocal; becaufe their Sides are all, refpectively, proportional; which is not the Cafe, in the Figures annexed. For, EF is not neceffarily in the fame Ratio to AC, as are the other Sides, AB to DE, and DF to BC. Neither can there, in fimilar Figures, be found the Antecedent of one Pair, and the Confequent of the other, in the fame Figure, unlefs they are congruous. (See Def. 43.)

The Sides of fimilar Figures are, therefore, directly proportional, and not reciprocally.

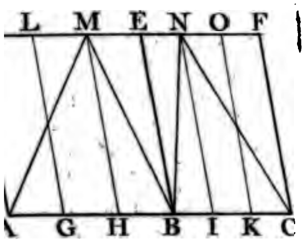
For the Definitions of Base and Altitude, fee Def. 45 and 46 in the general Introduction.

AXIOM. If two, or more, Figures are fimilar to the fame Figure, they are fimilar between themfelves.

THE

THEOREM I. 1 Euclid.

Parallelograms, or Triangles, having the ſame Altitude, are, to each other, in the ſame Ratio as their Baſes.



Let ADEB and BEFC be two Parallelograms, between the ſame Parallels, AC and DF.

I ſay, the Ratio of the Parallelogram AE to BF, (i. e. of their Areas) is, as AB to BC, their Baſes.

And, the Triangle AMB is to BNC, as the Baſe, AB is to BC.

Let AB be divided into any number of Parts, equal to each other, in G and H; and ſuppoſe BC divided into the ſame number of equal parts, in I and K (Poſt. 2. 5.)

Draw, GL, HM, &c. parallel to AD and BE.

DEM. Becauſe AG, GH, and HB are equal to each other; the Parallelograms AL, GM, and MB, are equal. - 18.1. And, the Pars. BN, NK, and KF, are alſo equal - ſame

Now, if AB=BC, the parts AG &c. are equal to BI, &c. If AB be greater than BC, the part AG is greater than BI; and, if AB be leſs than BC, AG is leſs than B I. Conſequently, the Parallelograms AL, GM &c. are, alſo, either equal to, greater, or leſs, than BN, NK, &c. as, AG, &c. is equal to, greater or leſs than BI, &c. For, Quantities are in the ſame Ratio, to each other, as their Equimultiples, or equal Parts. - - Ax. 8.

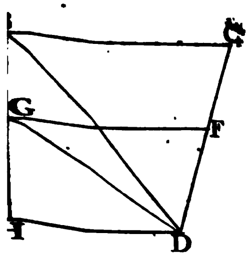
Now,

Now, AG:AB::BI:BC } in equal Ratio,-Ax. 7. 5.
And, Par. AL: AE::BN:BF }
Conf. Par. AL:BN::AG:BI, i.e.::AB:BC,(P.1.5.)-4.5.
But, Par. AL:BN::AE:BF. - - - fame.
Th. Par. AE:BF::AB:BC. - - Ax. 13. 5.

2nd. Becaufe Triangles are equal to half Parallelograms, having the fame Bafe and Altitude; - - 17. 1.
the Parallelogram AE : BF :: △. AMB : BNC - Ax. 8. 5.
and, Parallelogram, AE : BF :: AB : BC - above;
conf. the Triangle AMB : BNC :: AB : BC. - 13. 5.

COR. Parallelograms or Triangles having the fame or equal Bafes, have the fame Ratio to each other, as their Altitudes.

For, if their Bafes be confidered as their Altitudes, and their Altitudes as their Bafes, it is evident from the Theorem.



Let ABCD and AEFD be two Parallelograms, on the fame Bafe AD.

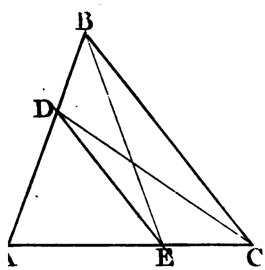
Draw the Perpendicular BH ;-(Prob. 7.)
BH and GH are their Altitudes. - Def. 46.

Then the Parallelogram AC : AF :: BH : GH.

For, fince all Parallelograms having the fame Bafe and Altitude are equal, the Parallelograms, AC, AF, are equal to Rectangles, whofe Bafes are equal to AD, and their other Sides equal to BH and GH, refpectively. - 18. 1.
Wherefore, BH and GH being confidered as the Bafes of the Parallelograms AC and AF, and their Altitudes, AD; the Parallelogram AC : AF :: BH : GH - - Theo.
But, the Triangle ABD is half the Par. ABCD, } - 17.1.
and, the Triangle AGD is half the Par. AEFD, }
Conf. the Triangle ABD : AGD :: BH : GH. - Ax. 13. 5.

THEOREM II. 2 Euclid

If a Right Line be drawn, parallel to any Side of a Triangle; it will cut the other two Sides (produced if neceſſary) proportionally.



Firſt. In the Triangle ABC, let DE be drawn, parallel to BC; cuting the Sides AB and AC, in D and E. - (Prob. 5 —)

Then will AB and AC, be cut proportionally, in D and E.

i. e. AD to DB, as AE is to EC.

Draw BE and DC.

DEM. Becauſe DE is parallel to BC, DBE = EDC - 18. 1 —
Wh. the Triangle ADE : DBE :: ADE : EDC. - Ax. 4 5 —
But, the Triangle ADE : DBE :: AD : DB; and, the Triangle ADE : EDC :: AE : EC. } - Th. 1 —
Therefore, - - AD : DB :: AE : EC. - Ax. 13. 5 —

Caſe 2. If ADE be the given Triangle; let BC be drawn (without it) parallel to DE.

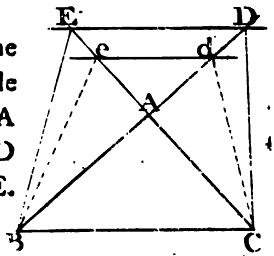
Then, if the Sides AD, AE be produced, cuting BC, in B and C; the Sides, AD and AE, of the Triangle, ADE, are proportional to the Segments, DB and EC, of thoſe Sides, produced to B and C.

For, as AD : DB :: AE : EC; - - - as above.
conſ. as AD : AE :: DB : EC. - - 4. 5.

COR. 1. If two Sides of a Triangle (AB, AC) are cut proportionally from the ſame Angle, (A, in D and E; ſo, that, AD : DB :: AE : EC) a Right Line (DE) joining the Points (D and E) will be parallel to the remaining Side of the Triangle. The Converſe.

For,

For, the Sides, AB, AC, being cut, as AD:DB :: AE:EC; and the Right Lines, BE, DC, drawn, as before;
Δ. ADE:DBE :: ADE:EDC; (1.) conſ. DBE=EDC.
Therefore, DE is parallel to BC; - - Cor. to 18. 1.



Caſe 3. If DE be drawn, beyond the Vertex A, parallel to BC, the Side oppoſite; and, if the Sides BA and CA be produced, till they cut DE, in D and E; then is AB to AD as AC to AE.

Join BE and DC; as before.
The Triangle ACD is equal to ABE.

DEM. DE is parallel to BC; wh. Δ DBE=DCE - 18. 1.
and, ADE is common to both; th. ACD=ABE - Ax. 7. 1.
Conſ. the Triangle ABC:ACD :: ABC:ABE - Ax. 4. 5.
But, - - - ABC:ACD :: AB : AD; } Theo. 1ſt.
and, - - - ABC:ABE :: AC : AE. }
Therefore, - - AC: AE :: AB : AD. - Ax. 13. 5.

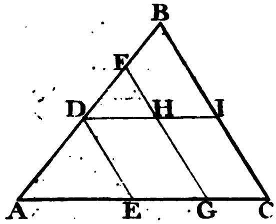
COR. 2. Two Right Lines (BD and CE) cuting each other (in A) between, or beyond, two parallel Lines (DE and BC) are cut proportionally, in that Point, and the Points in which they cut the Parallels.

For, AB:AD :: AC:AE, in the 3rd Caſe, directly; and alſo, in the other two, by compoſition. - Th. 6. 5.

COR. 3. A Right Line drawn parallel to the Side of a Triangle (in the firſt Caſe) cuts off a Triangle, ADE, ſimilar to the whole Triangle ABC; and, in the other two Caſes, the Triangles ADE and ABC are alſo ſimilar. (Def. 1.)

For, the Angle ADE=ABC, and AED=ACB - 4. 1.
and, the Angle A is either common, or vertical.

COR. 4. If several Lines be drawn, within a Triangle, parallel to any Side; the corresponding Segments of the other two Sides, cut by the parallel Lines, are proportional; each to the other, respectively.



Let DE and FG be parallel to BC, in the Triangle ABC.

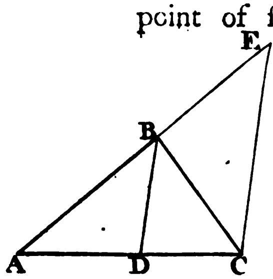
Then, AD : DF : FB :: AE : EG : GC.

For, the Segment AD : DF :: AE : EG. - Theorem.
Let DI be drawn parallel to AC; then, DF : FB :: DH : HI
But, EDHG and GHIC are Parallelograms, - by Constr.
wherefore, DH = EG, and HI = GC - - 15. 1.
Therefore, DF : FB :: EG : GC (as DH : HI) - Ax. 4. 5.
But, AD : DF :: AE : EG; th. AD : FB :: AE . GC - 9. 5.

THEOREM III. 3 Euclid.

If any Angle of a Triangle is bisected by a Right Line, cuting the opposite Side; the Segments of that Side, will be proportional to the two Sides of the Triangle, containing the Angle bisected.

And, conversely, if any Side of a Triangle be cut, in the proportion of the other two Sides; and if the greater Segment be contiguous to the greater Side; then will a Right Line, drawn from the point of section to the opposite Angle, bisect that Angle.



Let the Angle ABC, in the Triangle ABC be bisected by the Right Line BD, cutting the Side AC in D.

I say, AD is to DC, as AB is to BC.

DEM.

DEM. Produce AB indefinite; make BE=BC, and draw CE.
Then, becaufe BE=BC, the Angle BEC=BCE, - 9.1.
(for, the Triangle CBE is Ifofceles)
and the Angle ABC (which is external)=BEC+BCE-10.1.
Now, the Angle ABD = DBC, by Conftruction;
and BCE=BEC; confequently, DBC=BCE. - Ax. 3.1.
But, the Angles DBC, BCE, are alternate; and fince they are equal; confequently, BD is parallel to CE - 4.1.
Wherefore, in the Triangle AEC, the Sides AC, AE, are cut proportionally, by the Right Line BD;
i. e. AD : DC :: AB : BE - - - - Th. 2.
But, BE = BC (Con.) th. AD : DC :: AB : BC. - Ax. 4.5.

2nd. In the Triangle ABC, let the Side AC be fo cut, in D, that the Segments, AD, DC, are in the Ratio of AB to BC.
Then, a Right Line, BD, bifects the Angle ABC (the fame Conftruction remaining as before).

Now, BE=BC (Con.) and, AD:DC::AB:BC - Hyp.
wh. AD:DC::AB:BE; conf. BD is parallel to CE - 2.
and confequently, the Angle DBC= BCE - 4.1.
But, becaufe (in the Triangle CBE) BC = BE, - Con.
the Angle BCE = BEC; i. e. AEC. - - 9.1.
and, the Angle ABC = BCE + BEC. - - 10.1.
But, the Angle DBC = BCE; and ABD = AEC. - 4.1.
Therefore, ABD is equal to DBC. - - Ax. 3.1.

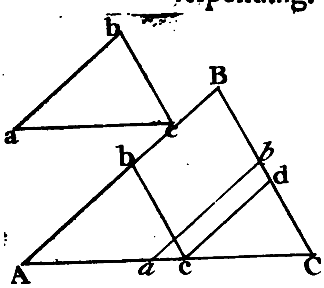
N. B. This Theorem and the foregoing are of extenfive utility, and are particularly ufeful in Perfpective.

The third Cafe of Theo. 2, and the following Corollary, contain the whole Subftance of practical rectilinear Perfpective.

THE

THEOREM IV. 4 Euclid.

In equiangular Triangles, the Sides containing equal Angles are proportional; and, the Sides which subtend equal Angles are homologous, or corresponding.



Let ABC, and abc, be equiangular Triangles, having the Angle a, equal to the Angle A, b equal B, and c equal C.

I say, the Side a b, is in proportion to A B, as a c is to A C, and, as b c to BC.

Dem. Suppose the Angle a applied to, or laid upon the Angle A, and a b upon AB. - - - Post. 5. 1.
Then, because the Angle b a c = B A C, a c will fall on AC, and b c will cut the Triangle ABC, in b c.
But, the Angle ABC = A b c; and are internal & opposite;
wh. bc is parallel to BC (4.1.) conf. Ab : bB : : Ac : cC -
Wh. Ab : Ab + bB (equal AB) : : Ac : Ac + cC (eq. AC) - 6.
Therefore, as a b : AB : : ac : AC.

Again; suppose the Triangle a b c applied to the Angle C of the Triangle ABC; (as *abC*) then will the Side a b (i. e. *a b*) be parallel to AB. - - - Cor. 1. Th. 2
The rest is as before; i. e. *b*C : BC : : *a*C : AC.
But *a*C = Ac, &c. and, Ac : AC : : Ab : AB, - above
consequently, *b*C (i. e. bc) : BC : : Ab (i. e. ab) : AB.

It is thus demonstrated, according to Euclid.

Le

4

Let Abc and cdC be equiangular Triangles, fo applied, at the Angle c, that the two Sides, Ac and cC, are in a Right Line.

Produce Ab and Cd, meeting in B.

Then, becaufe the Angle Acb = ACd, & Ccd = CAb, AB is parallel to cd; and, CB parallel to cb - 4.1.
wherefore, cbBd is a Parallelogram; - Def. 33.
confequently, bB = cd, and Bd = bc. - - 15.1.
But, Ab : bB :: Ac : cC; - - Th. 2.
wherefore, Ab : cd (equal bB) :: Ac : cC; }
alfo, as Ac : cC :: bc (equal Bd) : dC } Ax. 4 5.
confequently, as Ab : cd :: bc : dC. - - - 13. 5.

Cor. 1. Triangles, having all their Sides directly proportional, are equiangular. The Converfe; needs no proof.

Cor. 2. Triangles, being equiangular, or having all their Sides directly proportional, are fimilar. - fee Def. 13

Confequently, if two Angles, of one Triangle, are equal, refpectively, to two Angles of another Triangle, the Triangles are fimilar; and their Sides proportional.

For, the two remaining Angles are equal - 10. 1.

Hence is deduced the 15th Problem, and alfo the 16th.

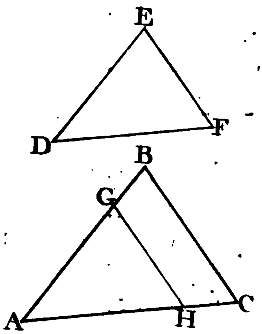
N. B. This Theorem is of univerfal application. It is the Criterion of Proportion in every branch of the Mathematics; for, wherever there is found equiangular Triangles, feeing that they are neceffarily fimilar, the Proportion of their correfponding Sides, is confequently analogous.

The firft Corollary, the Converfe of the Theorem, is the 5th Propofition of Euclid.

THE-

THEOREM V. 6 Euclid.

If, in two Triangles, an Angle of one be equal to an Angle of the other, and, if the Sides containing the equal Angles are directly proportional, the Triangles are similar.



In the Triangles ABC, DEF, let the Angle BAC be equal EDF; and, as AB is to DE, so let AC be to DF.

Let the Triangle DEF be applied to the Triangle ABC; the Angle D to the equal Angle A. - - Post. 5. 1.

Or, from the Angle A (equal D) take AG, equal DE, and AH equal DF, and join GH. - - - (Prob. 3. -)

Dem. Now, because AB : DE :: AC : DF; - (Hyp. .)
consequently, AB : AG :: AC : AH. - Ax. 5. 5.
wh. AB—AG (GB) : AG :: AC—AH (HC) : AH, - 7. 5.
i. e. GB : AG :: HC : AH; or, as AG : GB :: AH : HC - 5. —5.
wherefore, GH is parallel to BC; - - Cor. 1. 2. 6.
therefore, the Triangle AGH is similar to ABC. - C. —3.
But, the Triangles AGH, DEF are congruous - 8. — .1.
Therefore, the Triangle DEF is similar to ABC-Axiom.

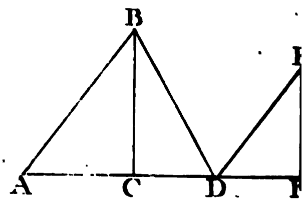
The 7th Proposition of Euclid, is of little or no consequence as it seldom, if ever, occurs; it is rather a critical remark, than a necessary Proposition.

THE

THEOREM VI.

In fimilar Triangles, the Perpendiculars, from equal Angles, are proportional to the correfponding Sides.

And, correfponding Sides are cut proportionally by the Perpendiculars.



Let, the Triangles ABD, DEG be fimilar, whofe correfponding Sides are AB and DE, AD and DG, BD and EG.

I fay, the Perpendiculars BC, EF, from the equal Angles, B and E, are proportional to the Sides.

DEM. For, the Triangles ABC, DEF, alfo CBD, FEG are fimilar; becaufe, the Angles, at C and F, are all Right (Def. 11.) therefore equal; - - Ax. 9. 1.
the Angle A = EDF, and BDC = G; by Hypothefis;
confequently, ABC = DEF, and CBD = FEG. - 10. 1.

Wherefore, as AB : DE :: BC : EF.
But, - - AB : DE :: AD : DG. } - Th. 4.
Therefore, - BC : EF :: AD : DG.
Alfo, - - BC : EF :: BD : EG. } - Ax. 13. 5.

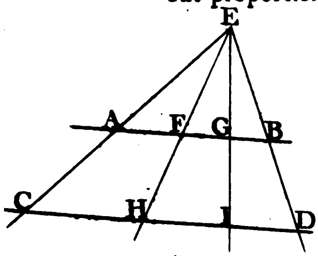
2nd. AD and DG are cut proportionally, in C and F, by the Perpendiculars BC and EF.

Becaufe AC : BC :: DF : EF;
and, - - BC : CD :: EF : FG } - - Th. 4.
conf. - as AC : CD :: DF : FG. - - - - 9. 5.

COR. If parallel Lines are cut by any number of Lines, proceeding from the fame Point, they will be cut proportionally.

 And

And, the Lines which proceed from a Point will alfo be cut proportionally, by the parallel Lines.

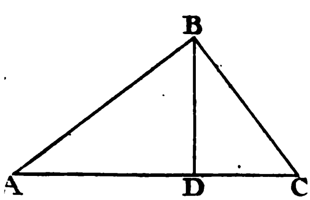


Let AB and CD be parallel Lines.
From any Point, E, at pleafure, if the Right Lines EA, EF, &c. are drawn, cuting the Parallels, in A and C, F and H &c; they will be cut proportionally, in thofe Points.

For, in the △s. AEF, CEH, the Angle EAF = ECH - 4— 1.
alfo, AEF is common to both Triangles;
confequently, the Triangles AEF and CEH are fimilar.
And, for the fame reafon, the Triangles FEG and HEI;
alfo, GFB and IED are, refpectively, fimilar.
Wherefore, EA : EC : : EF : EH : : EG : EI : : EB : ED.
confequently, AF : CH : : FG : HI, and as GB : ID.
Therefore, CH : HI : ID : : AF : FG : GB.
Alfo, EA : AC : : EF : FH : : EG : GI, and as EB : BD.
Hence is deduced the 36th, a moft ufeful Problem.

THEOREM VII. 8 Euclid 6.

In a right angled Triangle, if a Perpendicular be drawn, from the Right Angle to the Hypothenufe, it will divide that Triangle into two Triangles, which are fimilar to each other; and alfo to the whole Triangle.



In the Right angled Triangle, ABC, and, from the Right angle, B, draw the Perpendicular BD, to AC.

I fay, the Triangles ABD, DBC, are fimilar, to each other, and alfo to the whole Triangle, ABC.

DEM.

DEM. The Angle ADB=ABC (Con.) and BAD is common; consequently, the Angle ABD=BCD - - 10. 1.
Wherefore, the Triangles ABD, ABC are similar - 4.
Again; the Angle ADB=BDC (Def. 10.1)
and ABD=BCD (above) wherefore, BAD=DBC - 16. 1.
Consequently, the Triangle ABD is similar to DBC - 4.
But, the Triangle ABD is similar to ABC - proved
Therefore, the Triangle DBC is similar to ABC - Axiom.

COR. 1. In right angled Triangles, the Perpendicular (BD) is a mean Proportional between the Segments (AD & DC) of the Hypothenuse, made by the Perpendicular.

For, the Triangles ABD, DBC, being similar, the Side AD (of the Triangle ABD) is to BD (of the same) as the Side BD (of the Triangle D B C) is to D C (of the same)
i. e. AD : BD : : BD : DC. - - - - Th. 4.

From hence, and from Theorem 12 of the 3rd Book, are deduced the 30 and 31st Problems, in Practical Geometry; which are of most extensive utility.

COR. 2. In right angled Triangles, each Side containing the Right Angle, is a mean Proportional, between the adjoining Segment, and the whole Hypothenuse.

AB is a mean Proportioal between A D and AC;
and BC is a Mean, between D C and A C.

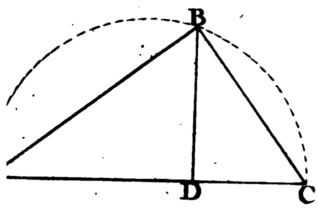
For, AD:AB (of the Tri. ABD) : : AB : AC, of the Tri. ABC }
And, AC:BC (of the Tri. ABC) : : BC : DC, of the Tri. DBC } - 4.

COR. 3. A Perpendicular, from any Point in the circumference of a Circle, to a Diameter, is a mean Proportional, between the two Segments of the Diameter, made by the Perpendicular.

For every Angle, touching the Circumference, in a Semicircle, is a Right one (12. 3.) consequently, the Diameter is the Hypothenuse of a right angled Triangle, inscribed in the Circle.

From hence, may easily be deduced the following Problems;

PROBLEM I. *The two Sides, or Legs, of a right angled Triangle being given, how to find the Hypothenuse, the Perpendicular, and the two Segments of the Hypothenuse, arithmetically.*



In the right angled Triangle, ABC, is given the two Legs, AB and BC.

1st. To find the Hypothenuse.

AB □ + BC □, = AC □ - 20.

Consequently, the Squares of AB and BC, being added together, the Square Root, of that Product, gives AC.

2nd. To find the Perpendicular, BD; AC : AB :: BC : BD - 4.

BD is, therefore, a fourth Proportional; AC, AB, and BC being the three Terms given, and BD is required - Def. 11. 5. Consequently, AB × BC = AC × BD - - (See Th. 9.) i. e. if AB be multiplied by BC, and that Product divided by AC (as in the Rule of Three) the Quotient arising is BD.

For, the Divisor multiplied into the Quotient is equal to the Dividend; i. e. AC multiplied into BD, is equal to AB multiplied into BC; by Theorem 9th.

Therefore, the Perpendicular, BD, is a fourth Proportional.

Hence, a Rectangle, under the two Legs of a right angled Triangle, is equal to a Rectangle under the Hypothenuse and the Perpendicular.

3rd. To find the greater Segment, AD; AC : AB :: AB : AD - 4.

Wherefore, AD, the greater Segment of the Hypothenuse, is a third proportional to AC and AB. - - See Def. 9. 5. Conf. AB □ = AC × AD; or, the Rect. CAD (see Cor. Th. 9.) i. e. AB multiplied by itself, and the Product divided by AC, gives AD.

Otherwise; BC : BD :: AB : AD.

Conf. AD is a fourth Ptoportional; and AB × BD ÷ BC = AD.

4th. To find the lesser Segment, DC; AC : BC :: BC : DC; or AB : BC :: BD : DC. Also, AD : BD :: BD : DC.

When either Segment, AD or DC, is found, the other is a third Proportional, between that Segment and the Perpendicular, BD; conf. AD × DC = BD □. But, if the Hypothenuse, AC, and either Segment be found, then, AC − AD is equal DC; and AC − DC = AD.

PROB. II. *The Perpendicular and either Side being given, how to find the other Side.*

Let AB and BD be given.

Becaufe AB □ = AD □ + BD □; - - 20. 1.
confequently, AB □ − BD □ = AD □.
i. e. the fquare of BD being fubtracted from the fquare of AB, the fquare Root of the remainder gives AD.
Then, as AD : AB : : BD : BC; wh. BC is a fourth Proportional.
Conf. AB × BD ÷ AD = BC. For AB × BD = AD × BC.

PROB. III. *The Perpendicular and either Segment being given, to find the other Segment, and the two Sides.*

Let AD be the given Segment.

Then, as AD : BD : : BD : DC. Wh. DC is a third Proportional.
Conf. BD multiplied into itfelf, and that Product divided by AD, gives DC. For AD × DC = BD □.
If DC had been given, it is juft the reverfe.

2nd. AD □ + DB □ = AB □. And BD □ + DC □ = BC □ - 20. 1.
Wherefore, if the fquare of AD be added to the fquare of BD, the fquare Root of that Sum gives the Side AB.

Again; AB being found, BC is a fourth Proportional.
For, AD : BD : : AB : BC. Wherefore, AB × BD ÷ AD = BC.
i. e. the fum of AB multiplied by BD, divided by AD gives BC.

PROB. IV. *The two Segments* (AD *and* DC) *being given, to find the Perpendicular* (BD) &c.

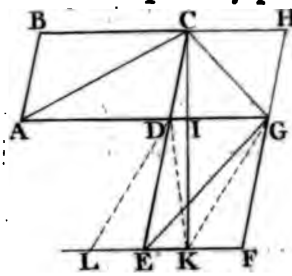
BD is a mean Proportional between the Segments AD & DC.
Wherefore, as AD : BD : : BD : DC;
confequently, AD × DC = BD □. - - (fee Cor. Th. 9.)
i. e. If AD be multiplied by DC, the fquare Root of that Product, is BD.
Hence, the Sides, AB and BC, may be found, as in Problem 3.

Thefe Problems, deduced from this extraordinary and extenfive Theorem, are extremely ufeful in Perfpective; to find Vanifhing Points and their Diftances, &c.

THEO-

THEOREM VIII. 14&15 Eucl. revers.

Parallelograms, having equal Angles; or, Triang[l] having one Angle, in each, equal to one anothe[r] and the Sides, containing the equal Angles, rec[i]procally proportional, are equal.



Let ABCD and DEFG be Parallel[o]grams, having equal Angles; and as t[he] Side AD, of the one, is to DG, of t[he] other, so let DE, of the same, be to D[C] of the first.

Let the equal Angles, at D, be so plac[ed] together, that, the Side AD, of one Parall[el]ogram, and DG of the other, are, in one Right Lin[e] then will DE and DC be also in one Right Line. - 2.

Produce BC and FG, meeting in H, forming a P[a]rallelogram CDGH.

DEM. Then, the Par. BD is to DH, as AD is to DG } T
and, the Parallelm. DF is to DH, as ED is to DC }
But, AD : DG : : ED : DC; by Hypothesis;
wherefore, the Parallelograms, BD and DF, have equal Ratio to the same Parallelogram, DH.
Therefore, the Parallelogram ABCD=DEFG - Ax.

2nd. Having drawn the Diagonals AC and EG; the [Tri]angle ACD is equal to DEG; - - Ax.
for, each is equal to half the Parallelogram BD or DF-
Or (CG being drawn) the Tri. ACD : DCG : : AD : [DG]
and, - - - the Triangle DEG : DCG : : ED : [DC]
But, AD : DG : : ED : DC (Hyp.) conf. Tri. ACD=D[EG]
for, they have an equal Ratio to the Triangle DCG.

Co[r]

Cor. 1. Equal Parallelograms having equal Angles, or Triangles having one Angle in each alſo equal, have their Sides about the equal Angles reciprocally proportional.

This, being the converſe of the Theorem is manifeſt, from it; for, by the ſame conſtruction, it is thus reverſed.

Becauſe, the Parallelogram BD is equal to DF, they have an equal Ratio to the Parallelogram DH - - Ax. 4. 5.
But, Par. BD:DH::AD:DG; and, Par. DF:DH::ED:DC - 1.
Therefore, as AD : DG :: ED : DC - - - - - - Ax. 13. 5.
Alſo, the Triangle ACD is equal to half the Par. BD; and DEG is equal to half DF (17. 1.) conſequently, ACD=DEG.-Ax. 4. 1.
and, their Sides, AD, DC; DE, DG, are reciprocally proportional.

Cor. 2. Parallelograms, or Triangles, having their Baſes and Altitudes reciprocally proportional, are equal.

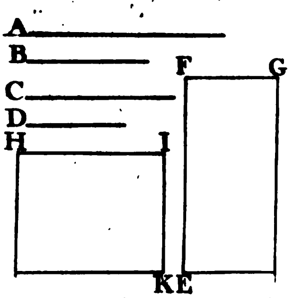
For, whether their Angles be equal or not, the Parallelograms are equal to Rectangles having the ſame Baſe and Altitude - 18. 1.
Conſequently, if their Baſes (AD and EF) and their Altitudes (CI and IK) are reciprocally proportional, the Parallelograms, BD and DF, or the Triangles, ACD, DEG, are equal - Th. 1.

For, the Perpendiculars (i. e. the Altitudes) of Triangles and Parallelograms, having equal Angles, are in the ſame Ratio as their Sides; i. e. CI : IK :: CD : DE - - - Th. 6, or 2.
But, their Baſes, EF and AD, are in the ſame Ratio.
Therefore, CI : IK :: EF, or KL (equal DG) : AD ; - Ax. 13. 5.
and conſequently, whether the Angles, at D, are equal or not,
the Parallelogram ABCD=DGKL (equal DEFG.) } - 18. 1.
Alſo, the Triangle DKG=ADC (equal DEG.) }

THEO-

THEOREM IX. 16 Euclid.

If four Right Lines are Proportionals, the Rectangle under the two Extremes (i. e. the greateſt and the leaſt) is equal to the Rectangle contained under the two Means.

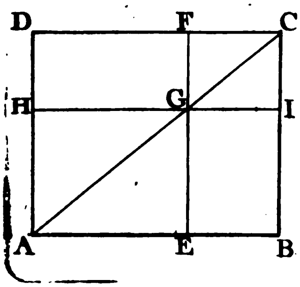


Let the four Right Lines, A, B, C, D, be proportional; and as A is to B, ſo let C be to D.

Then will the Rectangle under A and D, be equal to that under B and C.

Conſtruct the Rectangles EG and HK, making EF equal A, and FG equal D; alſo, let HI be equal C, and IK equal B. (Prob. 18.)

Dem. Then, becauſe EG and HK are Rectangles, conſequently their Angles are all equal. (Def. 34.) But, as EF : IK :: IH : FG; i. e. as A : B :: C : D; wherefore, their Sides are reciprocally proportional. Therefore, the Rect. EG, is equal to the Rect. HK.



Otherwiſe:

Take AB equal to the greateſt of the four proportional Lines, and, at either Extreme, B, make a Right Angle, ABC.

Make BC equal to the greateſt of the Means; and AE equal to the other.

Compleat the Rectangle ABCD, and draw the Diagonal, AC. - - (Prob. 18.)

Thro' E draw EF, parallel to BC; and through G, where it cuts the Diagonal, draw HI parallel to AB. (Prob. 5.)

Dem.

DEM. Now, because AB, BC, and AE are equal to three of the Proportionals, EG is equal to the fourth;
for, as AB is to BC so is AE to EG. - - Th. 4.
And, since the Complements, DG, GB, are equal - 19. 1.
if HE be added to both, the Rect. ADFE = AHIB - Ax. 6. 1.
But, AHIB is a Rect. under the two Extremes. } - Con.
And, ADFE is a Rect. under the two Means. }
Therefore, the Rectangle under the two Extremes, of four proportional Lines, is equal to a Rectangle under the Means.

Corollary. The 17th Proposition of Euclid.

If three Right Lines are Proportionals, the Rectangle under the two Extremes, is equal to the Square of the Mean.

If the two middle Terms, B and C, had been equal, the Rectangle HK would be a Square. (Def. 35.) See Fig. 1.
But, as EF : IK :: IH : FG; and IH is equal to IK. - Sup.
Therefore, the Rectangle under the Extremes, of three proportional Lines, is equal to the Square of the Mean.

Or, if (by the 2nd Construction) AE had been equal to BC, the Rectangle ADFE would be a Square.
Conf. it would be equal to the Rect. AHIB under the two Extremes. For, as AB : AE (eq. BC Sup.) :: AE : EG, eq. BI.

From hence, and Prop. 19. 1. is deduced the last method (Prob. 32) for finding a fourth Proportional.

A third Proportional may also be found, by the same; viz. by constructing a Square on either of the given Lines, which is to be the Mean.

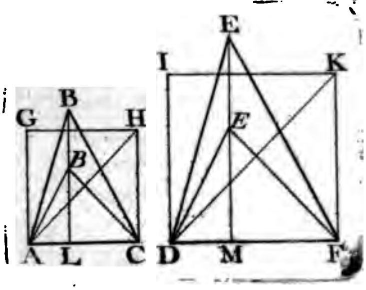
From which Constructions, it is obvious, is deduced that most excellent, Golden Rule, or Rule of Proportion, in Arithmetic.

N. B. If a Right Line be so divided in two Parts, that the Rectangle, under the whole Line and one Segment, is equal to the Square of the other; it is divided in extreme and mean Proportion in that Point.

For, the whole Line, the greater Segment, and the less, are in continual Ratio; consequently, they are three Proportionals.

THEOREM X.

Similar Triangles are proportional to the Squar[es of] their correſponding Sides.



Let ABC, DEF be ſimilar Tria[ngles] having the Angles at A and D equal the Angle B equal E, and C equal F

Then, Squares conſtructed, on c[orreſ]ponding Sides, AC and DF, or any have the ſame Ratio to each other, r[eſpec]tively, as the Triangles, ABC and D[EF].

Deſcribe the Squares AGHC and DIKF, on the AC and DF, which are homologous; being o[pp.] equal Angles, B and E.

Draw the Diagonals AH, and DK; alſo the Per[pendi]culars BL, and EM, to the Sides AC, and DE.

DEM. The △ABC : AHC :: BL : HC (eq. AC, Con) } C.
Alſo, the △DEF : DKF :: EM : KF, equal DF. }
But, - as BL : AC :: EM : DF; - -
wherefore, △ABC : AHC :: DEF : DKF - Ax.
conſ. - △ABC : DEF :: AHC : DKF - - Th
But, △AHC : □AGHC :: △DKF : □DIKF - 17. 1. &A
Wh. △ ABC : □ AGHC :: △ DEF : □DIKF -]
Th. △ABC : DEF :: □ AGHC : DIKF - Th
That is, the △ ABC : DEF :: AC□ : DF□. Q.

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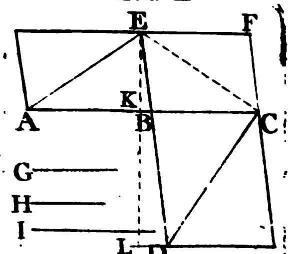
THEOREM XI. 23 Euclid.

Equiangular Parallelograms are, to one another, in a Ratio which is compounded of the Ratio of their Sides.



Let AE and DC be equiangular Parallelograms.

I ſay, the Ratio of the Parallelogram AE to DC, is equal to the compounded Ratio of their Sides. i. e. of the Ratio of AB to BC, and of EB to BD.

Let the equal Angles, at B, be ſo placed together, that, the Sides, AB, and BC, alſo DB and BE, are in a Right Line; and produce the oppoſite Sides, meeting at F, forming a Parallelogram BEFC.

Take any Right Line, G, at pleaſure; and, as BC is to AB, make H to G;
alſo, as DB is to BE, make I to H. (Prob. 32.)

Dem. Now, the Ratio of G to H is the ſame as AB to BC; and, the Ratio of H to I is equal to that of EB to BD.-Con. But, the Ratio of G to I, is compounded of the Ratios of G to H, and H to I (Def. 21. 5.) conſequently, the Ratio of G to I is that, compounded of the Ratios of the Sides; to which, theſe are reſpectively equal.

But, the Par. AE:EC::AB:BC; i.e. as G to H. }
And, the Par. EC:DC::EB:BD; i.e. as H to I. } - Th. 1.
Wh. ſince the Par. AE:EC::G:H; and Par. EC:DC::H:I.
The Parallelogram, AE: Par. DC::G:I. - Th. 9. 5.
But, G is to I, in the compounded Ratio of the Sides.
Therefore, the Par. AE is to Par. DC in the Ratio, which is compounded of the Ratio of their Sides. Q. E. D.

Cor. 1. Triangles, having one Angle in each, equal to one another, have that Ratio between them, which is compounded of the Ratio of the Sides, containing equal Angles.

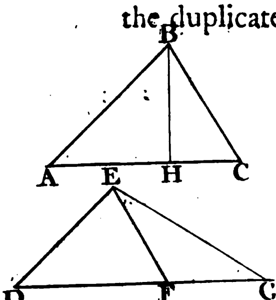
For, the Triangle AEB, BCD, having equal Angles, at B, and drawing EC, the Demonstration is in the Theorem.

Cor. 2. Parallelograms, having equal Angles, or Triangles, which have one Angle in each, equal to to one another, have that proportion, to each other, as the Rectangles under the Sides containing the equal Angles.

For, the Sides, containing equal Angles, have the same proportion, to each other, as the Perpendiculars, EK, KL.- Th. 6. Consequently, Rectangles under the Bases and Perpendiculars, are in the same Ratio to each other, as Rectangles under the Bases and adjoining Sides, containing equal Angles, viz. in the Ratio which is compounded of their Sides; by Theorem.

THEOREM XII. 19 Euclid.

The proportion of similar Triangles, to each other, is the duplicate Ratio of their corresponding Sides.



Let the Triangles ABC, DEF be similar and alike situated; i. e. let the Angle A be equal D, B equal E, and C equal F;

The Side AB corresponds with DE, BC with EF, and AC with DF. Produce DF.

Take DG a third Proportional, to DF and AC, and draw EG;

i. e. make DG to AC as AC is to DF; (Prob. 31.)

Then, the Triangle ABC is to DEF, as DG to DF.

Dem. Because the Triangles ABC, DEF are similar, consequently, AB : DE :: AC : DF. - - Th. 4.

But, DG : AC :: AC : DF (Con) consf. as AB : DE :: DG : AC.

Wherefore, in the Triangles ABC, DEG, the Sides AB and AC, DE and DG, which are about the equal Angles (A and D) are reciprocally proportional.

consequently, the Triangle DEG = ABC. - Th. 8.

But, the Triangle DEF:DEG::DF:DG - - Th.1.
Th. the Triangle DEF:ABC::DF:DG - Ax. 4. 5.
and, by inverſion, ABC:DEF::DG:DF. - Th. 5. 5.
i. e. in a duplicate Ratio of their correſponding Sides.

Or, if AH be taken a third proportional to AC and DF; it will then be, as AB:DE::DF:AH, i.e.::AC:DF-Con.
Conſequently, the Triangle ABH = DEF, - Th. 8.
Wherefore, the Tri. DEF:ABC::ABH:ABC.- Ax 4.5.
i. e. as AH is to AC (Th. 1.) viz. in a duplicate Ratio.

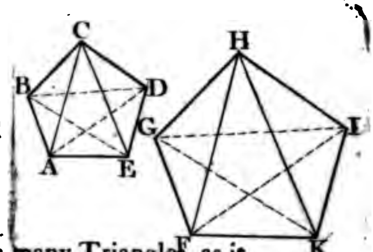
COR. If three Right Lines are Proportionals, the Squares of the firſt to the ſecond are as the firſt Line is to the third.

In the Tri. DEF, ABC, and DEG, DF:AC:DG ∺ - Con.
But, the Triangle DEF:ABC::DF:DG. - - Theo.
And the Triangle DEF:ABC::DF □ :AC □. - - 10.
Therefore, as DF:DG (the firſt to the third) ::DF □ :AC □.

THEOREM XIII. 20 Euclid.

Similar Quadrilaterals, and Poligons, are, by drawing Diagonals, divided into Triangles, equal in Number; the correſponding Triangles are ſimilar to each other, reſpectively; and proportional to the Poligons.

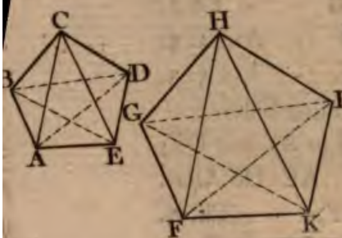
Alſo, the proportion of the Poligons, to each other, is in the duplicate Ratio of their correſponding Sides.



Let ABCDE and FGHIK be ſimilar Poligons.

Draw the Diagonals AC,CE, and FH, HK, &c. from any correſponding Angles, C and H, to the oppoſite.

DEM. Every Poligon is divided into as many Triangles, as it has Sides, wanting two; by drawing all the Diagonals which can be drawn, from any Angle; as CA, CE, or HF, HK; conſequently, the number of Triangles, in each, are equal; ſeeing the Poligons have an equal number of Sides.



2nd. Becauſe the Poligons are ſimilar, they are conſequently equiangular (Def. 1.) and, being ſimilarly poſited, the Angles $A = F$, $B = G$, and $C = H$ &c.

Then, becauſe $AB : BC :: FG : GH$ and the Angle $B = G$, the Triangles ABC, FGH are ſimilar. - - Th. 5.

And, becauſe the Angle $BAE = GFK$ and there is taken away, from each, equal Angles BAC, GFH, the remainder, $CAE = HFK$. - - Ax. 7. 1.

But, $AB:AC::FG:FH$, and $AB:AE::FG:FK$. conſequently, $AC : AE :: FH : FK$. - - 13.5. therefore, the Triangles ACE, FHK are ſimilar. Th. 5. And, becauſe $CD:DE::HI:IK$, and the Angles D and I are equal; the Triangles ECD, KHI are ſimilar. - ſame.

3rd. Becauſe the Triangles ABC and FGH are ſimilar, their Ratio is dupplicate of their correſponding Sides e. g. of AC to FH. - - - - Th. 12. And for the ſame reaſon, the Ratio of ACE to FHK is alſo duplicate of AC to FH; or, of EC to KH. - ſame. Alſo, the Ratio of ECD to KHI, is duplicate of EC to KH.

But, the Ratio of each Triangle, ABC, &c. to its correſponding Triangle, FGH, &c. is the ſame, viz. as AC to FH, or EC to KH, &c.

Therefore, as one Antecedent, ABC, is to one Conſequent, FGH, ſo is all the Antecedents ABC, ACE, and ECD, to all the Conſequents, FGH, FHK, and KHI, taken together, (2. 5.) i. e. as Poligon to Poligon. Q. E. D.

4th. The Ratio of ACE to FHK is duplicate of AE to FK. 12. But, the Ratio of Poligon to Poligon is equal ACE to FHK. Therefore, the Ratio of Poligon to Poligon, is duplicate of AE to FK, of ED to KI; or, of any other Sides, which are homologous.

COR. 1. All ordinate or regular Figures whatever, Triangles, Squares, or Poligons of any kind are, respectively, to each other, in the duplicate Ratio of their Sides, or Diagonals.

Because, all regular Figures, of the same Species, are similar. Also, Circles are, to each other, in the duplicate Ratio of their Diameters, &c.

Hence, in all similar Figures whatever, if the ratio of the homologous Sides be known, the proportion of the Figures is known.

For example; If the ratio of the Sides be as 2 to 3, their Areas are as 4 to 9. For, as $2 : 3 :: 3 : 4\frac{1}{2}$;
Therefore, as 2 is to $4\frac{1}{2}$, so is one Figure to the other;
which, in whole numbers, is as 4 to 9.

COR. 2. The Proportion of all similar Figures, whatever, are as the Squares of their corresponding Sides.

For, as the Triangle ABC or ACE, is to FGH, or FHK, so is Poligon to Poligon.
But, the Triangles, being similar, are, as the Squares of their homologous Sides; viz. as AB to FG, or AE to FK - Th. 10.
Therefore, as the Square of AB is to FG, or of AE to FK, &c.
so is Poligon to Poligon. - - - - Ax. 13. 5.

COR. 3. The Perimeters or Circuits of similar Figures are, to each other, as their corresponding Sides or Diagonals.

For, since each Side AB, BC, &c. has the same Ratio to its corresponding Side FG, GH, &c; also, as Side is to Side, so is Diagonal to Diagonal, AC to FH, &c; consequently, as any one Side, or Diagonal, is to its corresponding Side or Diagonal, so is the sum of all the Sides AB + BC + CD, &c. to the sum of all the Sides, FG + GH + HI, &c. - 2. 5.

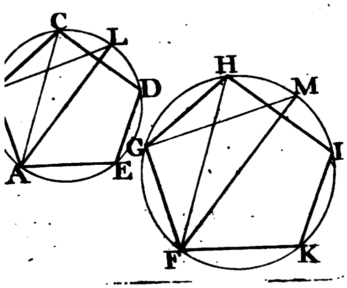
COR. 4. Any similar Figures whatever, described on the Mean and either Extreme of three proportional Lines, have the same Ratio to each other, as the two Extremes; i. e. as the first to the third.

For, they are to each other, in the duplicate Ratio of their corresponding Sides; by the Theorem.

From hence is deduced, an excellent Problem, (37) for finding the Side of any Figure whatever, similar to another, and in any Ratio.

THEOREM XIV.

The Perimeters of similar Poligons, inscribed in Circles, are, to each other, in the same Ratio as the Diameters; and their Areas as the Squares of the Diameters.



Let ABCDE and FGHIK be simila[r] Poligons, inscribed in Circles; whose Diameters are AL, and FM.

Then, as AL is to FM, so is the Perimeter ABCDE to FGHIK.

Also, as AL square, is to FM square, so is the Area of the Poligon ACE to FHK.

Join AC and BL, also FH and GM.

DEM. Then, the Angle BCA = BLA, and GHF = GMF - 10. — 3.
But, the Angle BCA = GHF (13) Th. BLA = GMF - Ax. 3 — 1.
and, the Angle ABL = FGM (for they are R. Angles) - 12. — 3.
therefore, the Triangles BLA, GMF are similar; - Th. — 4.
(for, the three Angles, are respectively equal)
Then, as AB : FG :: AL : FM. - - - - 4.
conf. as AB □ : FG □ :: AL □ : FM □. - 10.
But, as AB : FG :: Perimeter ABCDE : Per. FGHIK - 2. 5.
Th. the Diameter AL : FM :: ABCDE : FGHIK - Ax. 13. 5.
Also, as AB □ : FG □ :: the Area of ACE : FHK - Th. 10 & 13.
and conf. as AL □ : FM □ :: the Area of ACE : FHK - Ax. 13 5.

COR. The Circumferences of Circles are proportional to their Diameters, and their Areas to the Squares of their Diameters.

For, the Circumference of a Circle being considered as the Perimeter of a Poligon, of an infinite number of Sides, and, the Diameter may also be considered as a Diagonal; wherefore, the Demonstration evidently follows from the Theorem.

COR. 2.

COR. 2. Circumferences of Circles are proportional, to the Perimeters of ſimilar inſcribed, or circumſcribing, recti-linear Figures.

For the Perimeters, of all ſimilar right lined Figures, are in proportion to their correſponding Sides or Diagonals. - C. 3.13. And, the Circumferences of Circles, are as their Diameters.-Th.

COR. 3. The Areas of Circles are in proportion to that of any ſimilar inſcribed or circumſcribing Figures.

For, the areas of all ſimilar Figures, are as the duplicate ratio of their correſponding Sides or Diagonals; and, the areas of Circles are as the duplicate Ratio of their Diameters. - Th. 13, Wherefore, ſince the Diagonals of ſimilar Figures are in the ſame Ratio as their Sides, reſpectively; and alſo, as the Diameters of circumſcribing Circles; conſequently, the areas of Circles are, as the areas of ſimilarPoligons, inſcribed or circumſcribed.

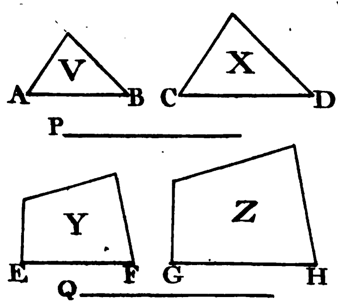
COR. 4. The area of a Circle is to that of any circumſcribed right lined Figure, as the Circumference of the Circle to the Perimeter of the circumſcribing Figure.

For, the area of a Circle is equal to a Triangle, whoſe Baſe is equal to the Circumference, and height, equal to the Radius. And, the Area of a circumſcribing Poligon, is equal to a Triangle, whoſe Baſe is equal to the Perimeter of the Poligon, and its height equal to the Radius of the inſcribed Circle. * Conſequently, the Areas of all circumſcribing Figures are as their Perimeters; which may be conſidered as the Baſes of Triangles of equal height. - - - - Th. 1. Therefore, the Area of a Circle is to that of a circumſcribed Poligon, as the Circumference of the Circle to the Perimeter of the Poligon.

* See, Theory of Menſuration; Article 7, and 8.

THEOREM XV. 22 Euclid.

If four Right Lines are Proportionals; similar Right-lined Figures, described on each Pair, will also be proportional.



Let the Right Lines AB, CD, EF, and GH be proportionals, and let AB be to CD, as EF to GH.

On AB and CD, let there be constructed similar Triangles, V&X; and, on EF and GH any other similar Figures, whatever, Y & Z.

Find P and Q each a third Proportional to AB and CD, and, to EF and GH, respectively. (Prob. 31.)

DEM. Now, the Triangle V is to X, as AB to P; and, the Trapezium Y is to Z, as EF to Q. } -Th. 1
But, as AB:CD::EF:GH; and, as CD:P::GH:Q -Co
wherefore, as AB:P::EF:Q (9.5.) Th. V:X::Y:Z.

It is also true, if similar Figures are constructed on the first and third, and others on the second and fourth, *of* four proportional Lines.

For, since V:X:Y:Z, cons. V:Y::X:Z. - Th. 4. 5.

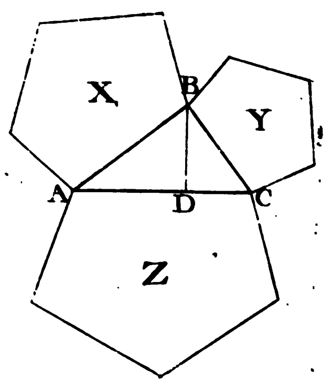
COR. Similar Figures being proportional to other similar Figures, their corresponding Sides are Proportionals.

The Converse, needs no proof; seeing it is but reversing the Premises.

THE-

THEOREM XVI. 31 Euclid.

Any ſimilar Figures whatever, deſcribed on the three Sides of a Right angled Triangle, and if they are made correſponding Sides of the Figures; then, that which is deſcribed on the Side ſubtending the Right Angle, will be equal to both the others, deſcribed on the Sides containing the Right Angle.



Let ABC be a Right angled Triangle; and, let X, Y, and Z be ſimilar, irregular Pentagons, deſcribed on the three Sides, AB, BC, and AC, of the Triangle; which are correſponding Sides of the Figures, X, Y, and Z.

I ſay, that the Figure Z, deſcribed on the Hypothenuſe, AC, is equal to both X and Y, deſcribed on the other two Sides.

Draw the Perpendicular BD.

DEM. In every Right angled Triangle (A B C), ſince AD : AB : : AB : AC; and DC : BC : : BC : AC - Cor. to 7, Conſequently, any Figure, deſcribed on AB, is to a ſimilar Figure, deſcribed on AC, as AD is to AC.
and a ſimilar Figure, deſcribed on BC, is to that deſcribed on AC, as DC is to AC. - - - 12 and 13.
Therefore, ſince Z : X : : AC : AD; and Z : Y : : AC : DC, Z will be to X + Y : : AC : AD + DC. - Cor. to 3. 5.
But, AC = AD + DC; therefore, Z = X + Y. Q. E. D.

 SHOL.

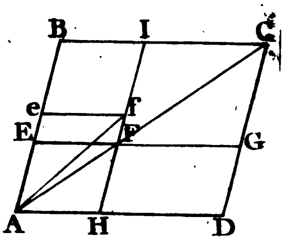
SHOL: *This Propoſition, by the Doctrine of Proportion, extends the famous Pythagorean Propoſition,* viz. *the 47th of the firſt Book of Euclid, and the 20th, of the firſt, of theſe Elements, univerſally.*

Seeing, by Th. 10. *of this 6th Book it is demonſtrated, that Triangles, and all ſimilar Figures whatever, are in proportion to the Squares of their correſponding Sides, the full and perfect demonſtration of this Propoſition neceſſarily follows from thence, and the 20th of the firſt; this Propoſition is therefore, in ſome meaſure, unneceſſary; but, for the particular beauty and elegance of it, I did not think proper to omit it.*

This Propoſition is alſo applicable to the Diagonals of Right lined Figures, as well as to their Sides; and, conſequently, to the Diameters of Circles. Wherefore, a Circle deſcribed on the Hypothenuſe of a Right angled Triangle, for its Diameter, is equal to the two Circles, whoſe Diameters are reſpectively equal to its Legs.

THEOREM XVII. 24 Euclid.

In every Parallelogram, thoſe which are about the ſame diagonal Diameter, are ſimilar to the whole Parallelogram and alſo to each other. And the two Complements have their Sides reciprocally proportional.



In the Parallelogram ABCD, let AEFH and FICG be Parallelograms, about the Diameter AC.

Then, the Parallelograms EH, IG, are ſimilar to the whole Parallelogram, BD, and to each other.

EG is parallel to AD, and BC; and HI, to AB and DC; by Conſtruction.

DEM.

EM. Now, becaufe E F is parallel to BC, the Sides AB and AC, of the Tri. ABC, are cut proportionally.
wherefore, AE : EB :: AF : FC - - Th. 2.
and conf. AE : AB :: AF : AC. - - - Conv. 6.5.
And, becaufe HF is parallel to DC, as AF : AC : : AH : AD.
wh.as AE : AB : : AH : AD; and, as AE : AH : : AB : AD-4.5
But, EF = AH, and BC = AD, &c. (15.1.) wherefore, the Sides of the Parallelograms A E F H, and A B C D, have all their Sides proportional.

But, the Angle EAH is common to both; and, the oppofite Angles of Parallelograms are equal - - 15. 1.
wherefore, E F H = B C D. - - - Ax. 3.
Alfo, the Angle AEF = ABC, and AHF = ADC. - 4.1.
therefore AEFH is fimilar to the whole Par. ABCD-Def. 1.

By the fame Reafoning, the Parellelogram FICG may be proved fimilar to A B C D.

For, CI : CB : : CG : CD; viz. as CF to CA - as above.
they are alfo equiangular; the Angle at C being common.
Confequently, the Par. FICG is fimilar to AEFH; being both fimilar to the whole Parallelogram, ABCD -Axiom.

d. The Compliments BF, FD are equal. - - 19. 1.
and they are equiangular, (15. 1.) for EFI = HFG - 2.1.
Wherefore, as EF : FG : : HF : FI, reciprocally - Th 8.

COROLLARY. 26 Propofition of Euclid.

Hence it is evident, that, if on any Angle of a Parallelogram, there be defcribed, or taken away, a leffer Parallelogram, fimilar, and a like fituated, they will have the fame common Diameter.

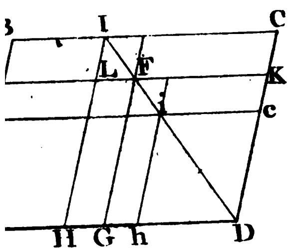
For, the Diameter AF or FC, is common with AC.

If any other Parallelogram A e f H, fimilar to ABCD, be defcribed at the Angle A, not alike fituated, they have not the fame Diameter AC.

THEO-

THEOREM XVIII. 27 Euclid.

If a Parallelogram be deſcribed on a Right Line, and from it there be taken away a Parallelogram, ſimilar, and alike ſituated, to one deſcribed on half the Line, equiangular to the firſt; that, which is deſcribed on the half Line, is greater than the remaining Parallelogram.



Let ABCD be a Parallelogram deſcribed on a given Right Line, AD; and, let AEFG be an equiangular Parallelogram, deſcribed on half the Line, AG; alſo, let HICD be taken away from the Parallelogram ABCD, ſimilar, and alike ſituated, to the Parallelogram AEFG.

I ſay, the Parallelogram AEFG is greater than the remaining Parallelogram ABIH.

Draw the Diameter ID, and produce EF to K.

Dem. Then, becauſe the Parallelograms, HC and GK, are about the ſame Diameter, ID, they are ſimilar. - 17.
wherefore, AEFG is ſimilar to HICD. - - Axiom.
Now, HF = FC (19. 1.) and BF = FC - Th. 1.
conſ. BF = HF (Ax. 3. 1.) and HF is greater than BL.
add AL, to both; and AF is alſo greater than AI. - 8.
Therefore, AEFG is greater than the remaining Parallelogram, ABIH, the defect of HICD.

Again,

Again. From the leſſer Parallelogram A b c D, let there be taken away the Parallelogram h i c D, ſimilar to AEFG, and ſituate alike.

Now, becauſe F D is a Diameter, in the Parallelogram GFKD, G i is equal to i K. - - - 19. 1.
Add h i c D, on both Sides; and G c = h K - Ax. 6. 1.
But, G b = G c. (Th. 1.) wh. G b = h K - - 3. 1.
Add G i, to both; and A i = GK − F i,
conſequently, AF (equal GK) is greater than A i.

Therefore, the Parallelogram AEFG, deſcribed on half the Line, AD, is greater than any other Parallelogram deſcribed on the whole Line; being deficient by a Parallelogram, ſimilar and alike ſituated to that which is deſcribed on half the Line. Q. E. D.

Few, if any, who have favoured the World with Treatiſes on Geometry, have taken notice of this Theorem, or the foregoing, (the 27 and the 24 of Euclid) except thoſe who have trod in his path without ſteping the leaſt aſide; indeed, it is ſo very obſcurely worded, that it is ſcarce intelligible; which, Mr. Stone has endeavoured to remedy, with little ſucceſs. He excuſes Euclid, by ſaying, that he could not have rendered it more clear, in ſo few words; and therefore, rather than appear tedious, gave it as it is; If the Propoſition, itſelf, be unintelligible, how are we to underſtand the Premiſes? Euclid has given ſome Propoſitions in more words. I am perſuaded that I have made it clearer than Mr. Stone, and in as few words as Euclid makes uſe of.

How far this Theorem may be of uſe in the Mathematics I do not pretend to ſay; but, when it is clearly underſtood, it will be found to contain ſomething extraordinary in it; inſomuch, that I could by no means diſpenſe with the omiſſion of it.

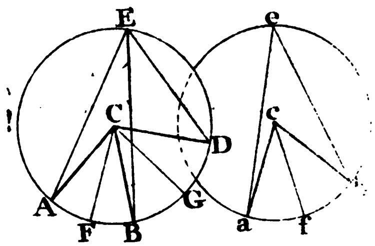
There are two Problems, following after, in Euclid, which are dependant on it; he alſo divides a Line in extreme and mean Ratio by it; in the 30th; it is certainly demonſtrable, when done, but how it is to be performed, in the operation, I cannot deviſe. (See, Prob. 11th. B. 2. or Pr. 35. Pr. Geometry.)

As I do not conceive the Problems to be at all uſeful to Mechanics, &c. I have not inſerted them, amongſt the reſt, in Practical Geometry.

The 32nd Prop. of Euclid is of little conſequence, and has nothing in it worthy of notice.

THEOREM XIX. 33 Euclid.

In the ſame or equal Circles, the Angles, whether at the Center or at the Circumference, are in proportion to the Arks on which they ſtand.



Firſt, in the Circle AED, the Angle ACB ſtanding on the Ark AB, is to the Angle BCD, on the Ark BD, as AB is to BD.

Let the Arks, AB and BD, be divided into any number of equal Parts, in F and G; and join CE, and CG.

DEM. Now, becauſe the Ark AF=FB, and BG=GD;
the Angle ACF=FCB, and BCG=GCD - C.2.9. 3.
wherefore, as the Ark AF:BG::FB:GD } in eq. } Ax.4.-5
and, as the Ang. ACF:BCG::FCB:GCD } Ratio }
For, whether AF be equal to, greater, or leſs than BG;
the Angle ACF is alſo equal to, greater, or leſs than BCG;
in the ſame Ratio. (Th. of Plane Angles.)
conſ. as the Ark AF:BG:: Angle ACF:BCG, &c.
Therefore, as AF + FB : BG + GD,
ſo is the Angle ACF + FCB : BCG + GCD. - 2.5.
i. e. as the Ark AB:BD, :: Angle ACB:BCD. Q.E.D.

But, the Angle ACB, at the Center, is double AEB, at the Circumference; and, BCD is double BED - 9.3.
wh. as the Angle ACB : BCD :: AEB : BED - Ax.8.5.
Therefore, as the Ark AB:BD, ſo is the Angle AEB:BED.

2nd. Becauſe, in the equal Circles, AEB, a e b, the Radii are equal; and equal Arks ſubtend equal Angles-C. 2. 9. 3. conſ. as Ark AB:a b :: Angle ACB: a c b; & AEB:a e b.

If AB and a b be divided into an equal number of Parts, the Demonſtration is the ſame as above.

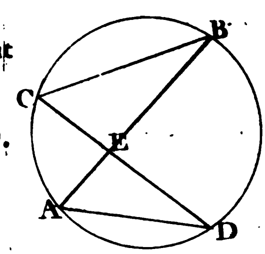
COR. Sectors of Circles, ACB, BCD, or a c b, are proportional to the Arks AB, BD, or a b.

For, each Sector may be conſidered as an aggregate of a number of Iſoſceles Triangles, of equal Baſes and equal Legs; as ACF, FCB, &c. conſequently, equal Triangles; and having equal Altitudes; each Sector, ACB, BCD, or a c b have, therefore, that Ratio to each other, as the Baſes of thoſe Triangles, i. e. as the Ark AB to BD, &c. - - - - Th. 1.

N. B. None of the following elegant, and ſome of them elementary, Theorems, are in Euclid; except the 34th; which is the 8th Propoſition of his thirteenth Book.

THEOREM XX.

If two Right Lines interſect, within a Circle, and are terminated by the Circumference, the Segments of thoſe Lines are reciprocally proportional.



The two Chords, AB and CD, cut each other in E.

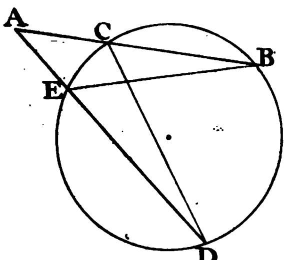
I ſay, that CE is to AE, as EB is to ED.

Join AD and CB.

DEM. Now, the Angles AED, CEB are equal. - - 2. 1.
and ABC=ADC; alſo BAD=BCD. - - 10. 3.
wherefore, the Triangles AED, CEB, are ſimilar. -Th. 4.
Therefore, CE:AE::EB:ED, by the ſame. Q. E. D.
Conſ. CE × ED = AE × EB; by 9. 6; alſo by 14. 3.

THEOREM XXI.

If two Right Lines are drawn, from the ſame Point without a Circle, to the oppoſite and concave part of the Circumference; they will have that proportion to each other, reciprocally, as their external Segments.



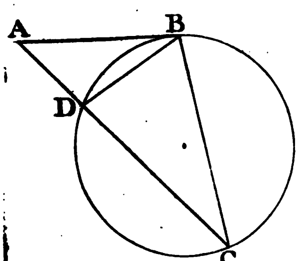
From the point A, draw AB and AD, at pleaſure, cuting the Circle EBD in the Points E, C, B, and D.

I ſay, that AB is to AD, as AE is to AC. Join EB and CD.

DEM. In the Tri. ABE, ADC, the Angle ABE=ADC·10.3
and the Angle A is common; wh. AEB=ACD - 10.1.
conſ. the Triangles ABE, ADC are ſimilar.
Wherefore, AB : AD :: AE : AC. - - - Th. 4.
And, AB×AC=AD×AE, by Th. 9. and, alſo by 16.3.

THEOREM XXII.

If from any Point without a Circle, two Right Lines are drawn, one touching the Circle, and the other cuting it; that which touches the Circle is a mean Proportional, between the whole Secant and the external Segment.



From any Point, A, draw AB, touching the Circle at B.
And, from the ſame Point draw AC, at pleaſure, cuting the Circle, in D and C.
I ſay, that AC is to AB, as AB to AD.
Join BC, and BD.

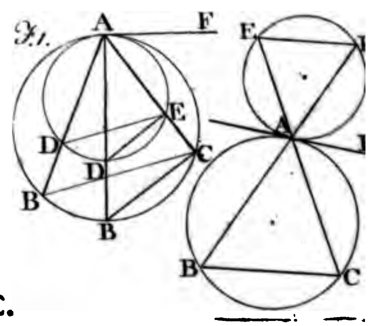
DEM.

DEM. The Triangles ACB, ABD, are ſimilar. - Th. 4.
For, the Angle ACB = ABD ; - - - 13. 3.
and BAC is common ; wherefore, ADB = ABC - 10. 1.
Therefore AC : AB :: AB : AD. Q. E. D.
And conſ. AC×AD=AB□. by Cor. Th. 9; alſo by 16. 3.

THEOREM XXIII.

If two Circles touch one another, either internally or externally, and if two Right Lines be drawn, from or through the point of Contact, cuting both Circles; thoſe Lines will be cut proportionally, by the Circumferences of the Circles.

In the Circles, ABC, ADE, and from, or through, the Point of Contact, A, draw AB and AC, cuting both Circumferences, in the Points B, C, D, and E.



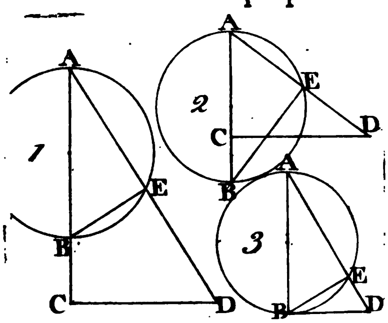
I ſay, that AB is to AC, as AD to AE.

Draw AF, a Tangent to the Circles, at the Point A; and join BC, and DE.

DEM. Then, the Angle ABC is equal to ADE ; - Ax. 3. 1.
For they are each equal to the Angle FAC. - Th. 13. 3.
And the Angle BAC (Fig. 1.) is common ;
In the ſecond, BAC, DAE, are vertical, th. equal - 2. 1.
therefore, the Triangles, ABC, ADE, are ſimilar,
and, conſequently, AB : AD :: AC : AE. - - Th. 4.

THEOREM XXIV.

If a Right Line be drawn perpendicular to the Diameter of a Circle, whether it be within or without the Circle, or touching the Circumference, and if any Right Line be drawn, from the fartheſt extreme of the Diameter, cuting the Circumference and the Perpendicular; that Line and the Diameter will be cut proportionally.



Let the Right Line CD be perpendicular to the Diameter AB.

Draw AD, at pleaſure, cuting the Circumference, in E, and the Perpendicular, in the Point D.

I ſay, that AB is to AE, as AD to AC. Join EB.

DEM. Then, becauſe A B is a Diameter, the Angle AEB, (being in a Semicircle) is a Right one; - - - 12. 3. and ACD is a Right one (Con.) th. equal to AEB -Ax.9. and, the Angle DAC is common; conſ. ADC = ABE-10.1. wherefore, the Triangles AEB, ACD are ſimilar. - 4. 6. Therefore, as AB:AE::AD:AC. Q. E. D.

In Fig. 3. DB and AC are the ſame; conſequently, AB is a mean Proportional between AE and AD.

COR. The Rectangle under any Line, AD, drawn from the extreme A, of the Diameter, to the Perpendicular, DC, and the part of it, AE, within the Circle, is equal to the Rectangle under the Diameter, AB, and the whole, AC, of the Diameter produced to the Perpendicular, (Fig. 1.) For, AE : AB : : AC : AD; th. they are four proportional Lines consequently, AE × AD = AB × AC. - - Th. 9.

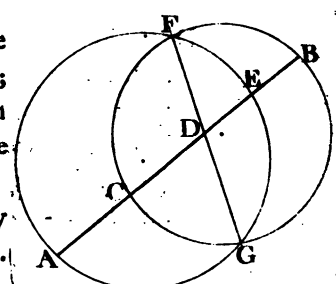
N. B. In Fig. 2nd, the Rectangle under the Means, AB and AC, is under the whole Diameter and a part of it.

In Fig. 3rd, since AE : AB : : AB : AD ; consequently, the Rectangle under AD and AE, is equal to the Square of the Diameter, AB. - Cor. to 9.

For, AD × ED = BD □ (16. 3.) } Therefore,
and, AB □ + BD □ = AD □ (20. 1.) } as above.

THEOREM XXV.

If two Circles cut each other, and a Right Line be drawn cuting both Circles, it will be cut proportionally, by the Circumferences, and a Right Line joining the points of intersection of the Circles.



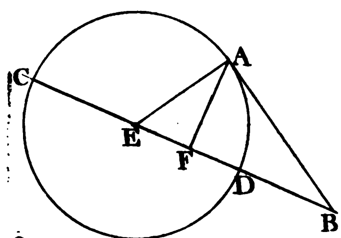
Join the Points, F and G, in which the two Circles, AFG and FBG, cut each other; and, let any Right Line, AB, cut both Circles, and the Right Line FG, in the Points A, C, D, E, and B.

I say, the Line AB is cut proportionally in those Points; viz. as AC : CD : : BE : ED.

DEM. For, in the Circle AFG, as AD : DF :: DG : DE }
And, in the Circle FBG, as DF : CD :: DB : DG } - 20.
wherefore, - - as AD : CD :: DB : DE - 10. 5.
Therefore, as AD — CD : CD : : DB — DE : DE - 7. 5.
That is, as AC : CD : : EB : ED. Q. E. D.

THEOREM XXVI.

If a Tangent to a Circle cuts any Diameter produced, and a Perpendicular be drawn, from the Point of Contact to that Diameter; the Segments of the Diameter, intercepted between the Center and the Perpendicular, the Circumference, and the Tangent are proportionals, i. e. the Radius is a mean Proportional betwixt the part, intercepted between the Center and the Perpendicular, and the Point in which the Tangent cuts the Diameter.



Let any Tangent, AB, cut the Diameter, CD produced, in B; and, let AF be perpendicular to CD; and E the Center.

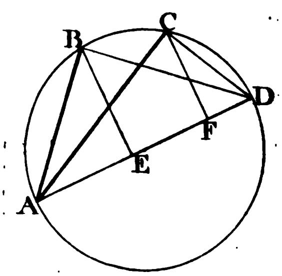
I ſay, that EF, ED, and EB are in continual Proportion.

DEM. For, becauſe EAB is a Right Angle, -	C. 3. 8. 3.
and, AF is perpendicular to EB; - -	Con.
the Triangles EAF, EAB are ſimilar. - - -	Th. 7.
Wherefore, as EF : EA :: EA : EB. - -	Cor. 2. 7
But, EA is equal to ED;	
therefore, - as EF : ED :: ED : EB. - -	Ax. 4.

TH

THEOREM XXVII.

The Squares of Chord Lines, drawn from one extreme of a Diameter, are in proportion to the Parts of the Diameter, intercepted between the Perpendiculars, from the other extremes of the Chords, to the Diameter.



AB and AC are two Chords, having a common extreme, A.

Draw a Diameter, AD, from the Point A; and, from the other Extremes, B and C, of the Chords, draw the Perpendiculars BE and CF, to the Diameter, cuting it in E and F.

I ſay, the Square of AB is to AC ſquare, as AE to AF.

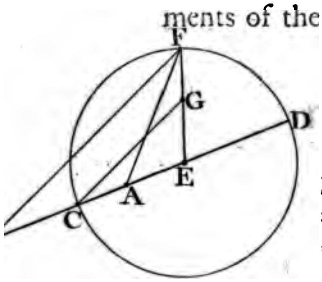
Draw BD and CD.

DEM. Now, BE is a mean Proportional between AE & ED; alſo, CF is a Mean, between AF and FD - Cor. to 7.6.
for ABD and ACD are Right Angles. - - - - 12. 3.
Wh. the Rect. AED=BE□; alſo AFD=CF□ -Cor. 9.6.
But AB□=BE□+AE□; and AC□=CF□+AF□,
conſ. AB□=AED+AE□; and AC□=AFD+AF□.
But, AED, i. e. AE×ED+AE□=AE×AD; and AFD, i. e. AF×FD+AF□=AF×AD; } 3.2.
wh. AB□=AE×AD; and AC□=AF×AD - Ax. 3. 1.
But, AE × AD : AF × AD : : AE : AF. - - Th. 1.
Therefore, AB□ : AC□ : : AE : AF. Q. E. D.

THEO-

THEOREM XXVIII.

If any Point, except the Center, be taken in a Diameter of a Circle, and if another Point (in the ſame Diameter, produced, and on the ſame ſide of the Center) be taken; the Diſtance of which, from the Center, is a third Proportional, to the diſtance between the Center and the firſt aſſumed Point, and the Radius; then will two Right Lines, drawn from thoſe Points to any point in the Circumference, be in the ſame Ratio, as the internal and external adjoining Segments of the Diameter.



Aſſume any Point, A, in the Diameter CD; in which Diameter, produced, take the Point B (from the Center, E) a third Proportional to EA and EC (Pr. 31.) and, to any Point, F, in the Circumference, let two Right Lines, AF and BF, be drawn.

I ſay, that AF is to FB, as AC to CB.

DEM. For, EA:ED (eq. EC) : :ED:EB - - - Con.
wherefore, EA : EA + ED : : ED : ED + EB. - - 6. 5.
i. e. EA:AD: :ED:BD; conſ. as EA:ED: : AD:BD - 4.5.
Now, EA : ED : : ED : EB; and EF = ED;
wh. EA : EF : : EF : EB. - - - - Ax. 4. 5.
therefore, the Triangles EFA, EFB are ſimilar - Th. 5. 6.
for, the Angle FEB is common, & the Sides are proportional.
Conſ. AF:BF: :EA:EF; i. e. : :EA:ED, or as AD:BD; that is, as AC to CB. Q. E. D.
For, EA:EC(eq ED): :EC:EB (i. e. as AD to BD)-Con.
conſequently, EA : EC : : EC—EA : EB—EC;
that is, - as EA : EC : : AC : CB. - - - - 3. 5.

Otherwiſe:

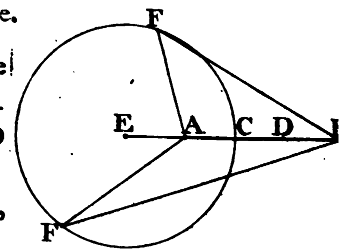
Otherwise:

Make EG equal EA, and draw CG.

Then, becaufe EA, EF, are equal EG, EC, refpectively, and the Angle at E is common, the Triangles AFE, ECG are congruous; and CG = AF. - - - 8. 1.
But, EA:EC::EC:EB (Hyp.) wh. EG:EC::EF:EB
conf. EG:EC::GF:CB (7.5.) wh. CG is paral. to BF-2.6.
and the Triangle ECG is fimilar to EBF - - - 5.
But, GF=AC (Ax. 7. 1.) for EF=EC; and EG=EA;
wherefore, as GF, i. e. AC, : CB, or, as EG : EC,
or, EF : EB :: CG, i. e. AF, : BF. Q. E. D.

COROLLARY. A PROBLEM.

If any Right Line, AB, be divided at pleafure, in C; to defcribe a Circle; to any Point in the Circumference of which, if Right Lines are drawn, from each extreme of the given Line, they fhall have the fame Ratio to each other, as the Segments of the given Line.



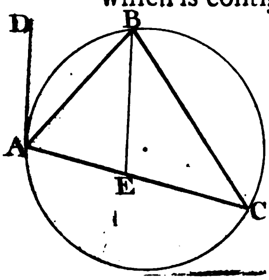
From the greater Segment, CB, take the lefs, AC; i. e. make CD equal AC.
Make AE a third Proportional, to BD and CD (equal AC)
With the Radius EC, on the Center E, defcribe a Circle;
and to any Point, F, in the Circumference, draw AF & BF.

Then, AF is to BF as AC is to CB. - Theorem.

Becaufe, EA:AC::AC (eq. CD) :DB - - Con.
Conf. EA:EA+AC (i.e. EC) ::AC:AC+DB (i.e CB)
i. e. as EA : EC :: AC : CB. - - Conv. to 6. 5.

THEOREM XXVIII.

If a Triangle be circumscribed by a Circle, and, a Tangent to the Circle be drawn, at any Angle of the Triangle; and, if a Right Line be drawn from the adjacent Angle, parallel to the Tangent, cuting the opposite Side; then, the Side between the Tangent and that Angle, will be a mean Proportional, between the opposite Side, and that Segment which is contiguous to the Tangent.



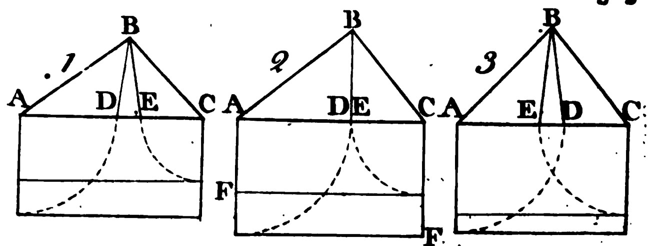
Let AD be a Tangent to the Circle ABCE, at the Point A, (the Angle of a Triangle, ABC, inscribed) and, let BE be drawn parallel to AD, cuting AC, in E.

I say, that AB is a mean Proportional, between AE and AC.

DEM. Now, B E is parallel to A D. - - Con.
wherefore, the Angle ABE = BAD - - - 4. 1.
But, the Ang. ACB = BAD (13 3) conf. ABE = ACB - Ax. 3. 1
and, in the $\triangle$s, ABE, ABC, the Angle BAC is common.
wherefore, those Triangles are similar.
Therefore, as AE : AB :: AB : AC. - Th. 4. 6.
consequently, AE, AB, and AC, are three Proportionals; and therefore, AB is a Mean.

COR. Hence it is manifest, that, what is proved, in Cor. 2. of the 7th, in respect of a right angled Triangle, is general; in every Scalene Triangle; on any Side of which, similar Triangles are constructed. e. g.

4 In



In the Triangle ABC, if ABD be made equal to the Angle C, and C B E equal A, the Angles A and C being common; the Triangles ABD, EBC, are similar to the whole Triangle ABC - - - C. 2 4. 6. and consequently, to each other. - - Axiom.

Wherefore, the square of AB is equal to a Rectangle, under the Side A C, and the Segment A D; and, the square of B C is equal to a Rectangle under AC and E C. - - - - - Cor. to Th. 9.

For because the Triangle A B D is similar to A B C; AD:AB::AB:AC; &, for the same reason, EC:BC::BC:AC Wherefore, AB and BC are, each, a mean Proportional; between the adjoining Segments of AC, and the whole, AC.

If the Triangle be right angled (Fig. 2.) the two Segments A D and E C are equal to the whole A C (for the Points D and E coincide; consequently, the two Squares of A B and B C are equal to the Square of AC.

i. e. A B square is equal to the Rectangle AF, } 20. 1.
and, BC square is equal to the Rectangle FC. - }

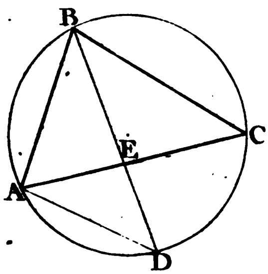
In Fig. 3rd, the Angle at B being acute; consequently, the two Segments AD and EC are greater than the whole AC; but they are, in each Case, subject to one general Demonstration.

When the Angles are all equal, the Square of each Side is equal to the Square of the other; for the Triangle will be equilateral.

THEOREM XXX.

If any Angle of a Triangle is bisected, and the bisecting Line cuts the opposite Side; the Rectangle under the two Sides, containing that Angle, is equal to the Square of the bisecting Line, added to the Rectangle under the Segments of the opposite Side.

Also, the Sides of the Triangle, the bisecting Line, and that Line produced till it cuts the Circumference of a circumscribing Circle, are Proportionals.



Bisect the Angle B, of the Triangle ABC, by the Right Line BE, cuting AC, in the Segments AE, EC.

Then, the Rectangle under the Sides AB and BC, is equal to the Square of BE, added to the Rectangle under AE and EC.

2nd. About the Triangle ABC, describe a Circle, ABCD. Produce BE to D, and join AD.
I say, that AB is to BD as BE is to BC.

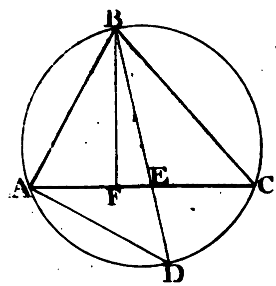
DEM. For, the Triangles ABD, EBC are similar - Cor.2.4.
because, the Angles ABD, EBC are equal; - - Con.
and, the Angle ADB is equal to ECB; - - 10. 3.
wherefore, - BAD is equal to BEC. - C. 5. 10. 1.
Th. AB : BD : : BE : BC (4) conf. AB×BC=BD×BE-9.6.
But, ▭DBE=DEB+EB□ (3.2.) & DEB=AEC -14.3.
Wherefore, the Rectangle DBE=AEC + EB□ -Ax.6.1.
Consequently, AB×BC =EB□ +AE × EC. Q. E. D.

The 2nd Part was proved in the fifth Line.

THE-

THEOREM XXXI.

The Rectangle, under any two Sides of a Triangle, is equal to a Rectangle contained under the Perpendicular to the other Side, from the opposite Angle, and the Diameter of a circumscribing Circle.



Describe the Circle A B C D about the Triangle ABC. - - (4. 4.)

Draw the Diameter BD; and BF perpendicular to A C.

Then, the Rectangle under AB and BC, is equal to the Rectangle under BF and BD. Draw AD.

DEM. Now, the Angle ADB = ACB; - - 10. 3.
and BAD is equal to BFC; - - - - Ax. 9. 1.
(for BFC is a R. Angle (Con.) and BAD is Right - 12. 3)
consequently, ABD is equal to FBC;
and the Triangles ABD, FBC are similar;
wherefore, as FB : BC :: AB : BD; - - 4. 6.
and therefore, FB×BD=AB×BC. Q. E. D. - 9. 6.

N. B. If the Angle, ABC, was bisected by the R. Line BD; the Rectangle under the Sides, AB, BC, is equal to a Rectangle under the Diameter BD and the Segment BE.

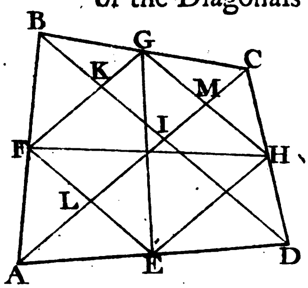
For, AB is to BD, as BE is to BC; by the 30th.

THE.

THEOREM XXXII.

If all the Sides of a Trapezium are bifected, and the Points of bifection, in contiguous Sides, are joined by Right Lines, the Quadrilateral formed by thofe Lines is a Parallelogram; and, it is equal to half the Trapezium.

Alfo, the Sum of the fquares of the Diagonals, of the Trapezium, is double the Sum of the fquares of the Diagonals of the Parallelogram.



Let the Sides of the Trapezium, A B C D, be bifected in the Points E, F, G, and H, and draw EF, FG, GH, and EH.

I fay, firft, that the Quadrilateral, E F G H, is a Parallelogram.

DEM. For, becaufe, in the Triangles ABC, ACD, the Sides A B, B C, and alfo A D, D C, are cut proportionally, in F and G, E and H, the Right Lines F G and E H are both parallel to the common Bafe AC; and, for the fame reafon, the Lines GH and FE are both parallel to BD; confequently, they are parallel between themfelves; therefore, E F G H is a Parallelogram. (Def. 33. 1.)

Secondly. It is equal to half the Trapezium ABCD.

For, in the Triangle ABC, becaufe FG is parallel to AC, and, becaufe AB and BC are bifected in F and G, B I is alfo bifected in K. - - - - 2. 6.

and,

and, for the ſame reaſon AI and I C are biſected in L & M; wherefore, LM is half AC; and conſequently, the Parallelogram L F G M (on half the Baſe A C, of the Triangle ABC, and half its Altitude) is equal to half the Triangle ABC - - Th. 18. 1. and Pr. 20.
for the ſame reaſon the Parallelogram L E H M = half the Triangle ADC.
conſ. the Par. LG + LH = half the Tri. ABC + ADC, i. e. the Par. F F G H is equal to half the Trap. A B C D.

Thirdly. Having drawn the Diagonals, EG and FH.

I ſay, that AC ſquare added to BD ſquare, is double the ſum of the Squares, of EG and FH.

For, the Squares of the Diagonals, EG and FH, are equal to the Squares of all the Sides EF, FG, GH, and E H. - - - - - - Th. 14. 2.
But EF and GH are each equal to half BD; and FG, EH, are each equal half AC; } - above.
wherefore, the four Squares, of EF, FG, GH, and EH, are equal to half the Squares of AC and BD - Cor. 4. 2.
and, they are equal to the ſquares of EG and FH - 2nd.
Therefore, the Squares of AC and BD, are double the Squares of EG and FH.

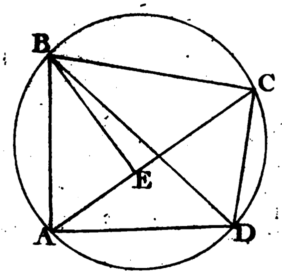
This Propoſition is, by Stone and ſome others, given in the ſecond Book; conſequently, there is a neceſſity for a Lemma, previous to it; which Lemma, is only a particular Caſe of the ſecond Propoſition of the ſixth Book; which cannot be made general without that Theorem.

The third Part of which he makes a ſeparate Propoſition, may be demonſtrated in the ſecond Book; but I did not think it of conſequence enough to make another Propoſition.

THE-

THEOREM XXXIII.

In every Quadrilateral inscribed in a Circle, the Rectangle contained under the Diagonals, is equal to two Rectangles under the opposite Sides.



In a Rectangle the thing is manifest - 20.1.

In the Trapezium ABCD, draw the Diagonals AC and BD.

I say, the Rectangle, under AC and BD, is equal to the two Rectangles, under AB and CD, BC and AD.

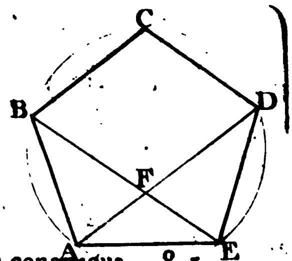
Make the Angle ABE equal to DBC. (Pr.4)

DEM. Now, ABE = DBC (Con) the Angle BAC = BDC - 10.3
wherefore, the Triangles ABE, DBC are similar - C.2.4.6.
Wh. as AB : BD : : AE : CD (4 6) & AB : AE : : BD : CD - 4.5.
Therefore, AB × CD = BD × AE - - - 9.6.

Again. The Triangle ABD is similar to EBC.
For, the Angle ABD = EBC (because, ABE = DBC (Con.) and the Angle EBD is common to both.) - Ax. 7. 1.
Also, the Ang. BDA = BCA (10.3.) conf. BAD = BEC - 10 1.
Wherefore, as BC : BD : : EC : AD - - - 4. 6.
consequently, BC × AD = BD × EC. - - - 9. 6.
But, - - AB × CD = BD × AE - proved above.
and AE, EC = AC; wh. BD × AE + BD × EC = BD × AC 1 2
Th. AB × CD + BC × AD = BD × AC. Q. E. D. - Ax. 3. 1.

THEOREM XXXIV. 8.13 Euclid.

'he Diagonal of a regular Pentagon is to its Side, as a Line divided in extreme and mean Proportion.



In the Pentagon A B C D E, let the two Diagonals, AD and BE, be drawn, cuting each other in F.

Then, A D is to A B, or A E, as A E is to AD — A E.

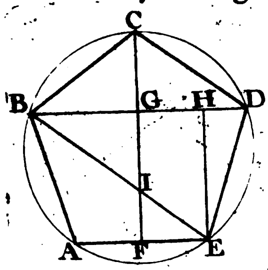
EM. The Triangles ABE, ADE, are congruous - 8. 1.
For, A B, A E, and E D are equal,
and the Angle B A E = A E D, - - - 8. 4.
wherefore, the Angle A B F = A E F = E A F - 9. 1.
conf. the Triangles ABE, AFE are fimilar. - C. 2. 4. 6.
Therefore, as BE:AE::AE:AF, equal FE. - 4. 6.
But, AE = BF; for DE = BC, and CD = AB - Hyp.
wherefore, BE is parallel to CD, and AD to BC - C. 10. 3.
confequently, B C D F is a Parallelogram;
wherefore, B F = CD (equal A B, equal A E). - 15. 1.
Therefore, as B E : B F :: B F : F E;
i. e. as B E : A E, or A B, :: A E : FE.
and confequently, A B, or A E, &c. is equal to the greater Segment of B E, or AD, divided in extreme and mean Proportion.

OR. 1. The Diagonal of a Pentagon is parallel to the oppofite Side.

Two Diagonals, cuting each other, are mutually divided in extreme and mean Proportion, in the point of interfection.

THEOREM XXXV.

The Diameter or Perpendicular, of a regular Pentagon, is divided in extreme and mean Proportion; by a Diagonal cuting it at Right Angles.



In the Pentagon A B C D E, let the Diameter C F be drawn, and the Diagonal B D, cuting C F in G, perpendicularly.

I say, that C F is divided in extreme and mean Proportion, in G.

Draw E H parallel to CF; also, the Diagonal BE, cuting the Perpendicular in I.

DEM. Now, because BE is parallel to CD, - C.1. 34.
the Angle C D B is equal to D B E - - - 4. 1.
and, CBD = CDB (9. 1.) wh. CBD = DBE - Ax. 3. 1.
But, the Angle CGB=BGI (Ax.9) wh. GI=GC - 11.1.
(for the Tri. BCG, BIG are congruous; BG being common)
Now, since EH is par. to FG (Con) & AE to BD; by the last;
FGHE is a Parallelogram; wherefore, EH=FG - 15.1.
And, because EH is par. to FG; EH:IG::BE:BI - 4.6.
(for the Triangles BGI, BHE are similar; by C. 3. 2.6.)

But, BE is to B I, as the greater Segment to the less, of a Line divided in extreme and mean Proportion. - 34.
(for, BI=BC) FG=EH; also, CG = GI, proved above;
wherefore, FG : GC :: BE : BI; i. e. as BI : IE;
and therefore, CF is divided in extreme and mean Ratio, in the Point G, where it is cut by the Diagonal BD.

COR. The Diagonal BE, cuts, FG, the greater Segment of CF, in extreme and mean Proportion; at I.

For BD is par. to AE; conf. the Tri. GBI, IEF are similar;
wherefore, as B I : IE :: G I : IF - - - - Th. 4.
But, BI:IE::BE:BI(eq. AB) therefore, FG:GI::GI:IF.

Hence, if a Right Line be divided in extreme and mean Proportion, and from either extreme of the greater Segment the meaſure of the leſs is ſet off, it will alſo be divided in the ſame Ratio; and the leſſer Segment of the firſt, will be the greater Segment of the other. e. g.

Let AB be divided in extreme & mean Proportion, in C.

Then, if CD be made equal to CB, AC will be divided in the ſame Ratio; and CD is the greater Segment.

For, ſince AB : AC :: AC : CB, (Hyp.) it will be, as AC:CB::AB−AC:AC−CB, i.e. as AC:CB::CD:AD.

Alſo. If to the whole Line, divided in extreme and mean Proportion, there be added the greater Segment; then, the whole compounded Line is divided in the ſame Ratio, and the greater Segment, of the firſt, will be the leſſer.

Let AC be divided in extreme and mean Proportion, in D. Then, if CB be made equal CD, CB will be the leſſer Segment of the whole, AB, divided in the ſame Ratio.

For, ſince AC : CD :: CD : AD; it will be as AC:CD::AC+CD:CD+AD; i.e. as AC:CD::AB:AC.

Hence, if either Segment be given, the other may be found.

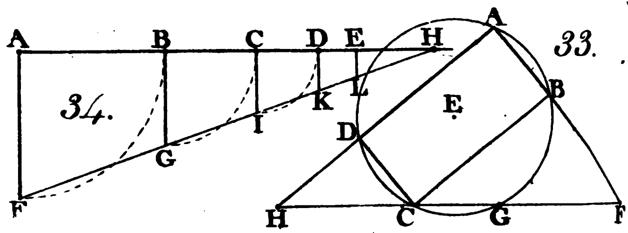
Let AC, the greater Segment, be given.

Divide AC in the Ratio required, in D. - (Pr. 35.)

Produce AC; make CB equal CD, and CB is the Segment ſought.

If CB, the leſſer Segment had been given. To find the greater; divide CB as before, in E; make CD equal CB, and AD equal CE. Then, AC is the greater Segment required.

 A De-



A Demonstration of the 33rd Problem.

FG is equal to CH (Con.) conf. FC is equal to GH - Ax.6.1.
wherefore, the Rectangles CFG, GHC, are equal - Ax.B.2.
But, AF × BF = CF × FG; also, AH × DH = GH × CH - C.16.3
consequently, the Rectangles AFB, AHD, are equal - Ax.3.1.
Wherefore, as AF : AH : : DH : BF, reciprocally - C.1.8.6.
But, as AF : AH : : BF : BC (Th.4) wh. BF : BC : : DH : BF - Ax.13.5

Again. It has been shewn (changing Terms) DH : BF : : BF : BC also, that BF : BC : : AF : AH ; i. e. as DC : DH. - Th. 4.
Therefore, the four Lines, DC, DH, BF, and BC are in continual Ratio ; and consequently, DH and BF are two Means, between DC and BC ; equal X & Z, by Construction.

A Demonstration of the 34th Problem.

First, AH : BH : : AB : BC; i. e. as AF : BG - - Th 4.
wherefore, AH : AB : : BH : BC - - - - 4 5.
conf. AH − AB (eq. BH) : AB : : BH − BC (eq CH) : BC - 7.5.
wherefore, AB : BH : : BC : CH - - - 5.5.
consequently, AB + BH : BH : : BC + CH : CH. - 6.5.
i. e. AH : BH : : BH : CH ; and as CH : DH, &c.
But, AH : BH : : AF : BG; and, BH : CH : : BG : CI, &c. - 4.6.
and AH, BH, CH, &c. are in continual Ratio ;
th. AF : BG : CI : DK, i. e. AB : BC : CD : DE, : : AH : BH, &c. ÷

But, AH : AF : : BH : BG, &c. and AH is greater than AF.
wherefore, DH is greater than DK, and EH than EL;
consequently, EL may still be taken from EH, *ad infinitum.*
th. AH is the whole sum of infinite Proportionals, to X and Z.

ELEMENTS OF GEOMETRY.

BOOK VII. The XI. of Euclid.

THE feventh Book, of thefe Elements, treats firft of the Elements or firft Principles of plane Solids, (i.e. Solids bounded by, or contained within Planes, only,) viz. of the Pofitions of Right Lines, and of Planes, to Planes; of the Sections of Planes by Planes, and of Right Lines by Planes.

Secondly, of folid Angles, generated by the interfections of Planes; their Conftruction and Principles inveftigated.

Thirdly, of Solids, contained within fix parallel Planes, only; their Affections, Proportions, and Properties, are fearched out, on which is founded the Theory of Menfuration of Solids.

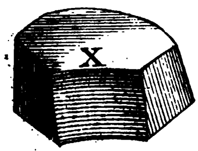
The Doctrine of Solids, contained in this Book and the following, is alfo of great ufe throughout the Mathematics; fome of which have their very exiftence in it. The Science of rectilinear Perfpective is wholly founded on the Sections of Planes by Planes, and, of Right Lines cut by a Plane, and by parallel Planes; infomuch that, without their affiftance, Perfpective would be a very imperfect Science; which, to the immortal Fame of Dr. Brook Taylor, is now eftablifhed on the moft permanent Principles.

Every Theorem, of the eleven firft, are more or lefs ufeful in Perfpective; the 5th and 7th are of great ufe; which, with the 8th and 9th contain the Effence of it, in Theory; as the 10th and 11th, of both Theory and Practice.

To enumerate every branch of the Mathematics, in which this feventh Book is particularly ufeful, is needlefs, was it in my power; the Reader may depend on it, that its Doctrine is, at the fame time, entertaining, and of extenfive utility.

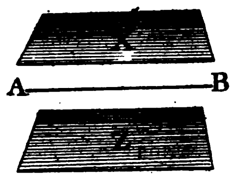
DEFINITIONS.

Def. I. A SOLID is a BODY, having length, breadth, and thickneſs; and is bounded by Planes, or curved Surfaces, or both. As X.



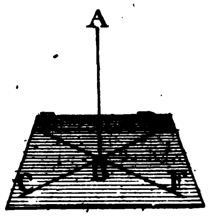
Def. II. A Plane (or Right Line) is parallel to another Plane, which, being produced infinitely, would never meet; but, are every where, equi-diſtant.

As the Plane X, to the Plane Z; or, the Right Line AB, to both X and Z.



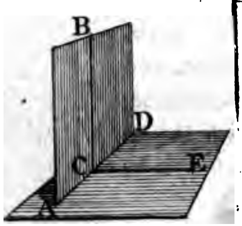
Def. III. A Right Line is perpendicular to a Plane, when it makes equal Angles with every Line drawn in the Plane, to or from that Point in which the Line cuts, or would cut, the Plane.

If AB makes Right Angles with the two Lines, CD and EF, paſſing through B, in which Point, AB cuts the Plane C E D F; and if the Lines, CD and EF, are both in that Plane; AB is perpendicular to the Plane.



Def. IV. A Plane, cuting another Plane, is perpendicular to the other, when it does not incline to the other on either Side.

If the Plane ABD cuts the Plane AED, in AD; and, if any Right Line, B C, drawn in one, be perpendicular to the other Plane (A E D) Thoſe Planes are perpendicular to each other.

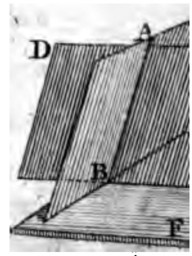


N.B. It is uſual, to define perpendicular Planes, by Right Lines being drawn perpendicular to their common Section; which, I preſume, is taking for granted that their common Section is a Right Line, and which, being granted, there would be no occaſion afterwards to demonſtrate it, in Propoſition 3rd.

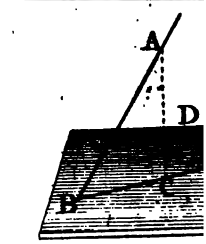
Let the young Geometrician take particular notice, that no regard is had to the poſition of the Plane, or Line, in reſpect of the Horizon, to which another Line or Plane is ſaid to be perpendicular, but only to their poſition in reſpect of each other, for, if they make Right Angles with each other, they are perpendicular, each to the other.

Def. V. If a Plane (or Right Line) be neither parallel nor perpendicular to another Plane, it is said to incline to the other Plane.

The inclination of one Plane to the other is the acute Angle ABC, made by the Section of another Plane, DEC; cuting both the inclined Planes, BE and BF, perpendicularly; i. e. at Right Angles.



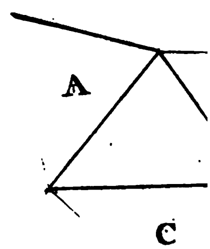
Also, the inclination of a Right Line, AB, to a Plane, BCD, is the Angle ABC made by another Line; drawn in the Plane BCD, from the Point B, in which the inclined Line, AB, cuts the Plane; and passing through another Point, C, in which a Perpendicular, AC, from any Point, A, in the inclined Line, cuts the Plane.



N. B. Two Planes have an equal inclination, to two other Planes when the Angles of their inclination are equal to one another.

Def. VI. A SOLID ANGLE is that which is made by more than two Plane Angles, applied close to each other, at the same Point, so, that two of them are not in the same Plane.

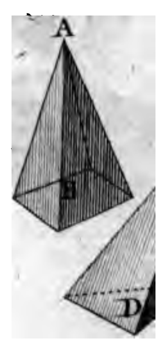
If the three plane Angles A, B, and C, be turned over, they will form a solid Angle, at the Point where their Vertices meet each other.



N. B. All the Plane Angles forming a solid Angle are, together, less than four Right Angles. (See Theorem 12.)
If three Plane Angles, form a solid Angle, any two must be greater than the third; or, three, greater than a fourth, &c. (See Theorem 13.)

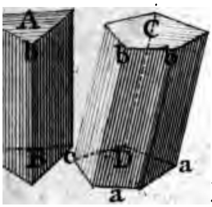
Def. VII. A PYRAMID is a SOLID, contained by any number of Planes more than three; of which, one (the Base) may have any number of Sides; all the other Planes are Triangles, meeting in a common Vertex.

As AB, or CD; A and C are their Vertices; B and D are their Bases.



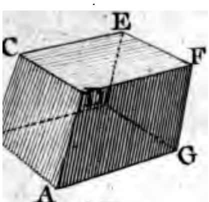
N. B. Not lefs than four Planes can form a Solid ; and that, muft neceffarily be a Pyramid, and the Planes are all Triangles.

Therefore a Pyramid may be faid to be the firft of Plane Solids; neverthelefs it is not the fimpleft.



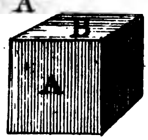
Def. VIII. A PRISM is a SOLID, contained within any number of Planes more than four; of which, two are parallel, equal, and fimilar Figures (of any number of Sides) all the other Planes are Parallelograms.

As AB ; or CD. A and B, C and D are the fimilar and equal Planes; the others, a b, b c, &c. are all Parallelograms.

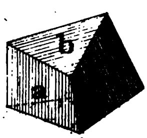


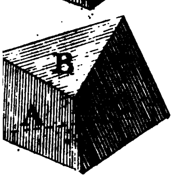
Def. IX. A PARALLELOPIPED is a PRISM, bounded by Six Planes; of which, the oppofite Planes are parallel.

If the Plane ABCD be parallel to DEFG, ABDG to DCEF, and ADFG to BCED; the Solid, ACF, is a PARALLELOPIPED.



Def. X. A CUBE is a PARALLELOPIPED, whofe Planes are all Squares. As A, B, C.





Def. XI. SIMILAR PLAIN SOLIDS are contained within an equal number of plane Figures, fimilarly fituated and fimilar one to another, refpectively, in each Solid.

The Angles are, alfo, neceffarily equal, each to the other, refpectively.

If the Plane a, be fimilar to A, b to B, and c to C ; and if all the other Planes, of which each Solid is conftructed, be alfo fimilar and fituate alike; then the Solid a b c is fimilar to A B C.

All CUBES are fimilar Solids.

All SOLIDS, contained within an equal number of equilateral and equiangular Planes, are fimilar.

---

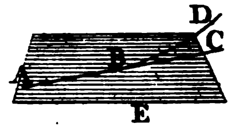
* A Prifm may be conceived to be generated, by the direct motion of any right lined Plane Figure; always parallel to its firft pofition, but not in a continuation of the Plane of the Figure.

A X I O M S.

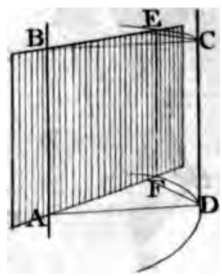
The firſt Propoſition of Euclid.

part of a Right Line is in the ſame Plane.

part of a Right Line cannot be in any another part of the Line out of that Plane. it was poſſible, a Right Line does not h a Plane in every Point; agreeable to Book 1ſt.



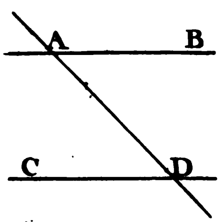
t AB, of a Right Line, ABC, be in the Plane whole Line is in the ſame Plane; for, the two and B, are in the Plane; and if any Point, in the Plane, ABD is not a Right Line; ſee- B is a Segment of the Right Line AC.



wo Right Lines, being parallel, are in lane, i. e. a Plane, may paſs through both

, being parallel, are in the Plane ABCD.

a Plane, ABEF, revolving on AB as an Axis. ince CD is parallel to AB, it is parallel to a g through AB; wherefore, the Perpendicu- l DF are equal (Def. 2.) and, conſequently, oint C coincides with E, D will coincide with whole Line CD is in the Plane, ABCD.



The ſeventh Propoſition of Euclid.

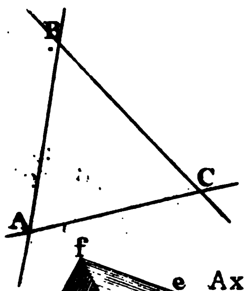
ight Lines, being parallel, and cut by ght Line, are all in the ſame Plane.

B be parallel to CD, they are in the Plane d) conſequently, the Points A and D, where em both, are in that Plane.

re, ſince there are two Points, A and D, of e AD, in a Plane, the whole Line is in that 1.) and, conſequently, AD is in the ſame the two parallel Lines, AB and CD.

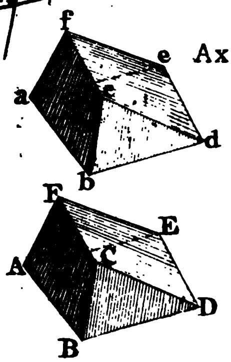
f two or more Lines, not parallel, are in the ſame re both cut by another Right Line, it is alſo in the ſame them. Conſequently, two Right Lines (AB and AD) other, are in the ſame Plane.

Ax. IV. The fecond Propofition of Euclid.



Three Right Lines cuting each other (not in the fame Point) are all in the fame Plane; wherefore, the three Sides of every right lined Triangle are in one Plane.

For, the two Right Lines, AB and AC, are in the fame Plane (3.) Conf. being both cut by another Right Line, (AC) feeing, the two Points, A and C, are in the fame Plane, the whole Line, AC, is alfo in that Plane - Ax. 1.



Ax. V. Solids, contained within an equal number of Planes, which are equal and fimilar to each other, two and two, (i. e. one in each Solid) refpectively; and being fimilarly fituated, in refpect of each other, and alfo of the whole; fuch Solids are equal in every refpect. As A C E and a c e.

If the Plane a b d e, be equal and fimilar to A B D E; a c, to AC, and c e to C E; alfo a f e, to A F E, and b c d, to B C D; then, the Solid a b c d e f, is equal in every refpect, to A B C D E F.

The eleventh Book of Elements is, in general, lefs underftood than any of the foregoing; not owing to any deficiency or imperfection in the Demonftrations, but very much fo to the imperfection of the Diagrams. It very rarely happens (and which never, yet, has happened) that a Mathematician, who had engaged in a Publication of the Elements of Geometry, was alfo, a competent Draftsman, in Perfpective; for want of which, the Schemes, which are intended to reprefent Planes cuting Planes, &c. are fo badly devifed, and worfe reprefented, that it is with the greateft difficulty a juft Idea of their meaning can be communicated; nor is it poffible, unlefs the Pupil has very acute Talents, or is affifted in his ftudy of them, by fome Apparatus, or an able Tutor.

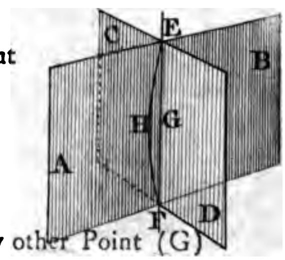
I flatter myfelf that I have removed that difficulty, which has hitherto been the greateft impediment, to a clear inveftigation of the Doctrine of Solids. Where it is neceffary, moveable Schemes are adapted for the purpofe; and, where they may be difpenfed with, the Diagrams are juftly, and perfpectively, reprefented; fo that, the Student muft not always expect (as in Plane Geometry) that, Angles which are faid to be right, or Lines parallel, &c. are really fo, in the Diagram; but, that they are, or would be fo, if fuch a Figure was really conftructed, as the Diagram reprefents. For, unlefs the Reader can conceive this, it will be very difficult for him to fuppofe one Line to be greater than another, when it is, abfolutely and evidently, lefs; which, from the Premifes given and the foregoing Elements, it is manifeft muft be greater.

Therefore it is, in ſome meaſure, neceſſary (by way of Poſtulate) to requeſt, that one Line be equal to another; or at right angles, i. e. perpendicular to another Line, or Plane, &c; that two Right Lines contain an Angle, equal to that which two other Lines contain; which, in the Diagram, is, perhaps, either greater or leſs. Theſe things muſt be granted and underſtood to be ſo, or no Demonſtration, of what is intended, can poſſibly follow.

I thought it neceſſary to appriſe the young Student of theſe preliminaries; otherwiſe (without a Tutor) he would, frequently, be perplexed and bewildered; and would, very probably, conclude, that there muſt be ſome miſtake, or error in the Diagram; for it is not eaſy, at firſt, to conceive, that one Line ſhould be perpendicular to another, when we ſee and know, that the Angles they make, are not Right ones; and, being acute, that another Angle, apparently contained within the other, and manifeſtly leſs, is a Right Angle; i. e. underſtood to be ſo.

THEOREM I. 3 Euclid.

The common Section of two Planes is a Right Line.



Let the two Planes, AB and CD, cut each other, in EF.

I ſay, that EF is a Right Line.

DEM. The Points E and F, and every other Point (G) in the Line EF, are in both Planes; for, the Line FG is in both Planes; ſeeing it is their common Section.-Ax. 1. Therefore, it is a Right Line; ſeeing that, a Plane agrees with a Right Line in every Point; - - Def. 6. 1. and, a curved Line cannot poſſibly be in two Planes.

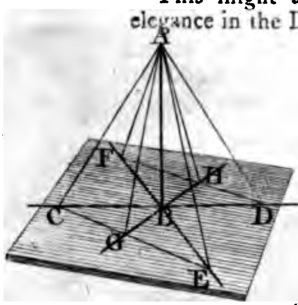
For, ſuppoſe the curve Line, EHF, drawn through the Points E and F, in the Plane ABC; if it was their common Section, it muſt alſo be in the Plane, CD; and conſequently it would not be a Plane, but a convex Surface.

This Theorem might paſs for an Axiom, being ſelf-evident.

THEOREM II. 4 Euclid.

If a Right Line ſtands at right angles, in the *Point* of interſection of two Right Lines; it will be perpendicular to the Plane, in which thoſe Lines are.

This might alſo paſs for an Axiom; but there is a peculiar elegance in the Demonſtration, which cannot be diſpenſed with.



Imagine the Right Line, AB, ſtanding at right angles to the two Right Lines, CD and EF, in the Point B of their common Section.

Then, AB will be perpendicular to the Plane CEDF, in which the two Lines, CD and EF, are ſituated.

Make BC equal to BD (at pleaſure) and imagine BE, BF, alſo equal to BC, BD.

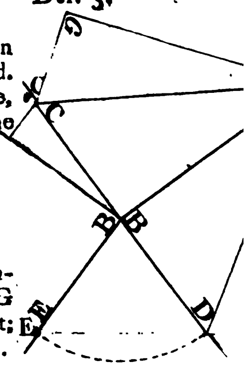
AB ſtanding perpendicular to them, imagine the Right Lines AC, AD, AE, and AF, drawn to any Point, A, in AB.

Join CE and DF; and, through B, draw any Right Line, GH, at pleaſure, cuting CE and FD, at G and H; and draw AG, AH.

Dem. Then, becauſe CB, BD, BE, and BF are all equal, and AB is common; alſo, the Angles ABC, ABD, ABE, ABF, are all ſuppoſed Right, therefore equal; the Right Lines AC, AD, AE, and AF, are all equal. - - 8. 1.
wh. the Angle ACE=AEC; and ADF=AFD - 9. 1.
And, becauſe the Angle CBE=DBF (2. 1.) and CB, BE are equal reſpectively, to BD, BF; CE=FD - 8. 1.
conſequently, the Angle BCE=BDF, - ſame.
Alſo, the Triangle CAE, being equilateral to FAD*, they are conſ. equiangular; & the Angle ACE=ADF-7. 1.

2 Now,

Now, the Angle BCE = BDF; and CBG = DBH - 2. 1.
and CB = BD; wh. the Triangle CBG = DBH; } - 11. 1.
and, consequently, CG = DH, and BG = BH. }
Wherefore, in the Triangles CAG, DAH; AC, CG, are respectively equal to AD, DH, and contain equal Angles; consequently, AG is equal to AH. - - 8. 1.
wh. the Triangles, AGB, BAH, are equilateral, to each other; (for BG, was proved, equal to BH, AG equal AH, and AB is common) consequently, the Angle ABG = ABH.
Therefore, AB is perpendicular to GH - Def. 10. 1.
and, consequently, to every Right Line, drawn through B, in the Plane CEDF; and therefore, AB is perpendicular to the Plane, CEDF. Q. E. D. - - Def. 3.



To assist the Imagination, I have added, Fig. 2. in which, if the Plane CAD be turned up (on CD) perpend. and, the Triangle ABE also vertical, till AB, in one, coincides with AB in the other; then, turning over the Triangles CA*G*, DA*H*, till *G* and *H* fall on G and turning back the Triangle AB*F*, till its Base into the Right Line EF; in which position may seen, what is represented in Figure 1.

The Line AB is perpendicular to the Plane CEDF.
For, CB, BE, BD, and BF are equal; and consequently AC, AE, AD, and AF are equal; also, AG is equal to AH, and the Angles ABG, ABH are Right; consequently AB is perpendicular to the Plane CEDF.

COROLLARY. Proposition V. of Euclid.

If three Right Lines meet in one Point, and a fourth stands at right angles to each, in that Point, the three Lines are in the same Plane.

If AB makes Right Angles with CB, and BE, and also with BG, they are necessarily in the same Plane - Def. 3.

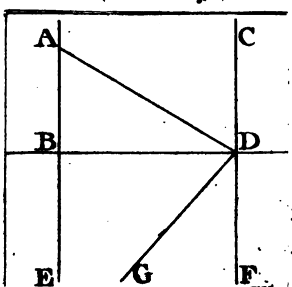
For, if ABG be less than a Right Angle, BG will be on this side the Plane, CHF; and, if it exceeds a Right Angle, BG will fall on the other Side. But, ABG is a Right Angle. Therefore, BG is in the same Plane with BC and BE.

* Triangles are equilateral to one another, when the Sides of one are equal, respectively, to the Sides of the other.

THEOREM III. 6 Euclid.

If two Right Lines be perpendicular to the ſame Plane, they are parallel to one another.

This might well paſs for an Axiom; for, it is evident, the two Lines may be in the ſame Plane, which is thus proved.



Let AB and CD be at right angles with the Plane B E F D.

I ſay, that AB is parallel to CD.

Draw BD, through the points, B and D, in which the Lines; AB & CD, cut the Plane. Alſo, draw BE and DF, perpend. to BD.

DEM. Becauſe AB and CD are both perpendicular to the Plane BGD, they are at right angles with every Line in that Plane, drawn through the Points B & D, reſpectively; therefore, to BE and DF (2.) and, conſequently, they don't incline to the Plane BGD, on any Side - Def. 3.
Wherefore, a Plane may paſs through both Lines - Def. 4.
But, the Angles ABD and CDB are Right - Def. 3.
Therefore, AB is parallel to CD - - - Th. 4. 1.

If the Plane BACD be turned up perpendicular, to BGD, the thing is manifeſt; ſeeing, the Lines AB and CD, make Right Angles with BE and DF, reſpectively.

According to Euclid (AD being drawn) the three Lines, AD, BD, and CD, are proved to be at right Angles with DG, ſuppoſed perpendicular to BD; but, the proceſs, though ſtrictly true, appears very lame and contradictory in the Figure.

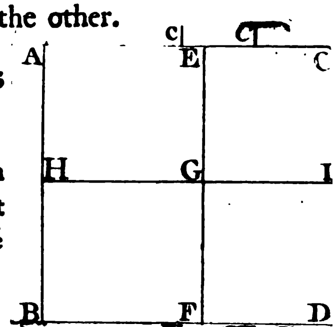
COROLLARY. The 8th Propoſition of Euclid.

If two Right Lines be parallel, and if one of them be at right angles with ſome Plane, the other Line is alſo at right angles with the ſame Plane.

This, being the Converſe of the Theorem, is manifeſt.

THEOREM IV. 9 Euclid.

Right Lines being parallel to the ſame[t] Right Line, which is not in the ſame Plane with them, are parallel amongſt themſelves; i. e. each to the other.



Let AB and CD be each parallel to EF; AB is parallel to CD.

In EF, take any Point, G, from which draw Right Lines, GH and GI, both at right angles with EF, and cuting the Lines AB and CD reſpectively.

DEM. By the Premiſes, EF is not in the ſame Plane with AB and CD; therefore, raiſe up EF, out of the Plane, but parallel to it, till CD coincides with *CD*, or any other Line, c d, parallel to EF.

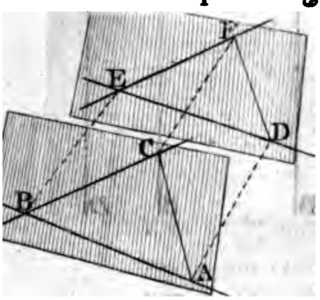
Then, becauſe GH and GI are at right angles with EF, EF is perpend. to a Plane paſſing through thoſe Lines - Th. 2
But AB and CD are both parallel to EF;
wherefore, they are alſo at right angles with the Plane HGI, paſſing through GH and GI - Cor. to Th. 3.
Therefore, they are parallel between themſelves; by Th. 3.

[t] When the three Lines are all in the ſame Plane, as in Theo. 5. Book 1ſt. it is manifeſtly an Axiom; and, in this Caſe, it amounts to little more; ſeeing that, a Plane may paſs through any two Right Lines, which are Parallel to one another. - - Ax. 2.

THE-

THEOREM V. 10 Euclid.

If two Right Lines, cuting or meeting each other, be refpectively parallel to two other Right Lines, alfo meeting or cuting one another, though not in the fame Plane with the firft, they fhall contain equal Angles.



Let the two Right Lines, AB and BC, be refpectively parallel to DE and EF; viz. AB to DE, and BC to EF; and, fuppofe they are not in the fame Plane.

I fay, the Angle ABC is equal to DEF.

Take BA equal to ED, BC equal to EF, and draw the Right Lines AD, BE, & CF.

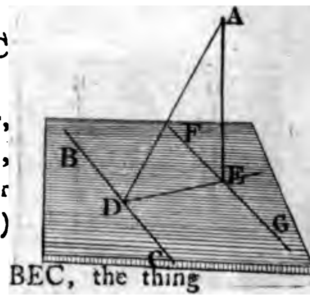
DEM. Becaufe AB is equal and parallel to DE, - Hyp.
AD is equal and parallel to BE - - Cor. to 15. 1.
and, for the fame reafon, BE is equal and parallel to CF;
wherefore, CF is equal and parallel to AD - Ax. 3. 1 & Th. 4.
Join AC and DF, by Right Lines.
Now, becaufe AD is equal and parallel to CF,
AC is equal (and alfo parallel) to DF; as above.
Then, in the Triangles ABC, DEF, the three Sides, AB, BC, and AC, of the one, are refpectively equal to DE, EF, and DF of the other;
wherefore, the Angles, of one, are alfo, refpectively, equal to the Angles of the other. - - - - 7. 1.
Therefore, the Angle ABC is equal to DEF;
being oppofite equal Sides, AC and DF. Q. E. D.

From the foregoing Theorems may be deduced the following Problems.

PROB-

PROBLEM I. 11 Euclid.

From any given Point, to draw a Right Line perpendicular to a Plane, in which that Point is not situated.*



Let A be the given Point, and BEC a Plane.

In the Plane BEC, draw, at pleasure, the Right Line BC, and, from A, draw the Right Line AD, perpendicular to BC. - - - (Prob. 7.)

If AD be perpendicular to rhe Plane BEC, the thing is already done; but, if it be not, proceed as follows.

From the Point D, draw DE, in the Plane BEC indefinite, also perpendicular to BC; (Prob. 6.) and from A, draw AE perpendicular to DE. (Prob. 7.)

I say, that AE is perpendicular to the Plane BEC.

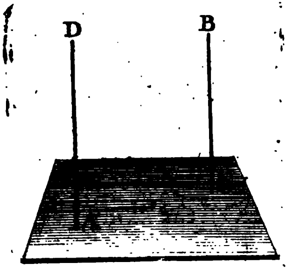
Through the Point E, draw FG, parallel to BC; (Pr. 5.) consequently, it is in the same Plane with BC. - Ax. 2. because the Point E is in the Plane.

DEM. Now, because AD & DE are both perpendicular to BC, BC is perp. to a Plane passing through AD & DE - Th. 2. But, FG is parallel to BC; by Construction; wherefore, FG is also perpendicular to the Plane ADE. Now, since AE is perpendicular to FG, and, also to DE, it is perp. to the Plane in which those Lines are situated - 2. Therefore, AE is perpendicular to the Plane BEC. Q.E.D.

* Professor Simson says, from a Point given above the Plane. I am not a little surprized at the expression, from one of his extraordinary Sagacity; because, the same thing holds true however the Plane be situated; or the Point in respect of the Plane, provided it be not in the Plane.

PROBLEM II. 12 Euclid.

From a Point given in a Plane, to draw a Right Line perpendicular to the Plane.



Let A be the given Point, in the Plane AC.

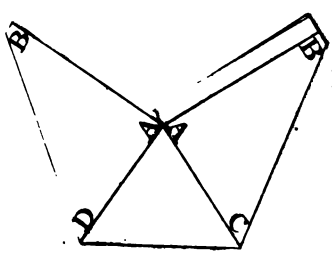
Assume any Point, B, out of the Plane; and, from that Point, draw BC perpendicular to the Plane; by the foregoing.

Then, from the Point A, draw AD, parallel to BC. - - (by Prob. 5. 1.)

∴ AD is perpendicular to the Plane AC.

DEM. Because BC is perpendicular to the Plane - (Prob. 1.) and AD is parallel to BC; by Construction; AD is perpendicular to the Plane AC. - - Th. 3.

I believe, that the manual operation, of this Problem, would be easier performed by two Right Angles. Thus



Let A be the given Point, at which a Perpendicular is required, to the Plane ACD.

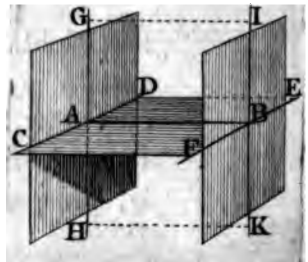
Apply a Right Angle, BAC, to the Point A, at pleasure, on AC; and, in any Angle (CAD) apply another R. Angle BAD. i. e. raise up the two right angled Triangles BAC, BAD, on AC and AD, till the Points, B, coincide. Then, AB is perpendicular to the Plane CAD. - Th. 2.

COR. Hence it is manifest, that there cannot be drawn two Lines, from the same Point, perpendicular to a Plane, and on the same Side.

THEO-

THEOREM VI. 14 Euclid.

Planes, to which the same Right Line is perpendicular, are parallel to one another.



Let the Right Line AB be perpendicular to both the Planes, CD and FE. I say, those Planes are parallel.

Imagine a Plane, C D E F, to pass through the Line AB, cuting both Planes, in CD, and EF; which Lines, necessarily pass through the Points A and B, in which the Line AB cuts those Planes, respectively. - Ax. 1. and Th. 1.

At the Points A and B, draw AG and BI, in the Planes CGDH and FIEK, perpendicular to CD and EF (Prob. 6.) and produce them to H and K.

DEM. Then, because G H and I K are perpendicular to CD and E F, and these Lines are in the Plane C D E F; G H and I K are perpendicular to that Plane; - Th. 2. (Because, A B being perpendicular to both the Planes, C G D H and F I E K, it is perpendicular to every Line drawn through the Points A and B; th. to GH & IK) wherefore, they are parallel to one another; - Th. 3. and, consequently, the two Planes cannot meet; being produced, either way, towards G and I, H and K - Ax. 1.

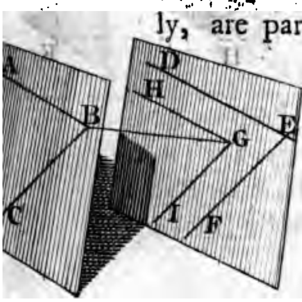
Again. Draw any other Right Line through A & B, in the Planes C G D H and F I E K, at right angles, with, or making equal Angles, and inclined the same way towards G H and I K; as C D and FE.

Then, because the Angle GAD = IBE, and AG is parallel to I B; and those Lines are perpendicular to A B; C D is parallel to E F. - - - - Ax. 10. 1. and consequently, the Planes CGDH and EIFK cannot meet, being produced towards D and E, or F and C.

Th. the Planes CGDH and FIEK are parallel - Def. 2.

THEOREM VII. 15 Euclid.

If two Right Lines, cuting or meeting each other, be parallel to two other Right Lines, alſo meeting each other, not in the ſame Plane with the firſt two; Planes paſſing through each two Lines, reſpectively, are parallel.



Let, the Right Lines AB, and BC, meeting at B, be parallel, reſpectively, to DE and EF, meeting at E, and not in the ſame Plane with AB, and BC.

The Planes ACB, DFE, drawn through thoſe Lines, are parallel.

From the Point B (in which AB meets BC) draw BG perpendicular to the Plane DFE (Pr. 1. 7.) and, through the Point G, in which BG cuts that Plane, draw, GH and GI, reſpectively, parallel to ED and EF. (Prob. 5. 1.)

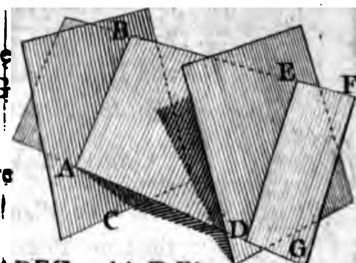
DEM. Then, becauſe BG is perpendicular to the Plane DEF GH, and GI make Right Angles with BG. - Th. 2.
But, AB is parallel to GH, and BC to GI; - - 5.
(for, they are reſpectively parallel to ED and EF)
And, BGH, BGI are Right Angles; - - Th. 2.
conſequently, ABG and CBG are Right Angles - 4. 1.
wherefore, BG is perpendicular to the Plane ABC; - 2. 7.
and, conſequently, the Plane ACB is parallel to DFE, in which the Lines AB, BC, and DE, EF are ſituated. } 6.

THE-

THEOREM VIII. 16 Euclid.

If two, or more, Planes, which are parallel, be cut by another Plane; their common Sections are parallel.

This Theorem might pass for an Axiom, or be deduced from the 6th, as a Corollary; but it is easily demonstrated.



Let the Planes ABC and DEG be parallel; and, imagine a Plane, ABFG cuting them, in AB and DE.

I say, the Sections, AB and DE, are parallel to one another.

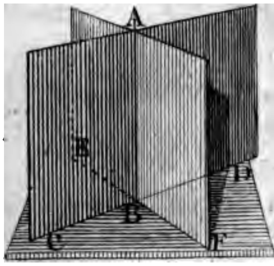
Dem. If AB (being in the same Plane, ABFG, with DE) be not parallel to DE, they will meet, if produced, towards one extreme on the other. - - Def. 7. 1.
But, AB is in the Plane ABC; and DE, in DEG - Th. 1.
And, the Plane ABC is parallel to DEG. - - Hyp.
Therefore, since both Lines are in the Plane ABFG, and are, also, respectively, in the Planes ABC, DEG, AB is parallel to DE; seeing they cannot go out of those Planes, and consequently can never meet. - Ax. 1.

THEOREM IX. 18 Euclid.

If a Right Line be perpendicular to a Plane; every Plane, which passes through that Line, is also perpendicular to the Plane.

Let AB be perpendicular to the Plane CEDF, and let any Plane CAD, or EAF, pass through the Line AB.

I say,



I ſay, the Planes, C A D and E A F, are perpendicular to C E D F.

DEM. Now, becauſe A B is perpendicular to the Plane C E F (by Hypotheſis) it does not incline to it on any Side - Def. 3.

But, the Line AB is in the Plane CAD; and it is alſo in the Plane E A F; (by Suppoſition). conſequently, thoſe Planes do not incline, on either Side, to the Plane CEF; ſeeing that, every Line, BC, BE, BD, and BF, make Right Angles with AB. - - - Th. 2.
Th. the Planes, CAD, EAF are perp. to CEF - Def. 4.

COROLLARY I. The 19th Propoſition of Euclid.

If two Planes, cuting each other, be perpendicular to the ſame Plane; their common Section is perpendicular to that Plane. The Converſe.

For, ſince the Planes CAD, EAF, cuting each other in the Line AB, are perpendicular to the Plane CEF, AB is perpendicular to CEF.
Becauſe, there cannot be drawn from the ſame Point, B, on the ſame ſide of the Plane CEF, two Right Lines, perpendicular to the ſame Plane. And AB is in both Planes - - Th. 1.
Therefore, AB, being in both Planes, CAD, EAF, is perpendicular to the Plane CEF. - - - Cor. to Prob. 2.

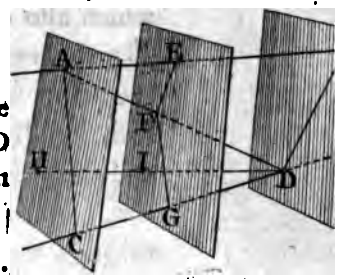
COROLLARY II. The 38th Propoſition of Euclid.

If from any Point in a Plane, which is perpendicular to another Plane, a Right Line be drawn, perpendicular to that other Plane; it will cut the other in the common Section of the two Planes.

For, ſince one Plane is perpendicular to the other (Hyp.) every Line drawn from any Point, in one, perpendicular to the other, is wholly in the firſt Plane; as AB.
Conſequently, ſince it cannot go out of that Plane, (Ax. 1.) it muſt neceſſarily cut the other Plane in their common Section.

THEOREM X. - 17 Euclid.

Right-Lines, being cut by parallel Planes, are cut proportionally.



Let the Right Lines, AB and CD, be cut by the Planes ACH, EG, and BD (being parallel between themselves) in the Points A, E, B, and C, G, D.

I say, as AE is to EB, so is CG to GD.

Join the Points A and C, B and D; and, AD being drawn diagonal-wise, cuting the Plane EG in F, draw EF and FG, joining the Points where they cut the Plane, EG.

DEM. Then, because AB, AD, and BD are three Right Lines, cuting one another, they are in the same Plane ABD; and ADC is also a Plane, by the same - Ax. 4.

Now, since the Planes BD and EG are parallel, and are both cut by the Plane ABD;
their common Sections, BD and EF are parallel; } Th. 8.
and, for the same reason, AC is parallel to FG. }
But, in the Triangle ABD, because EF is parallel to BD, the Sides AB, AD are cut, as AE : EB : : AF : FD - 2.6.
And, because FG is parallel to AC, as AF:FD::CG:GD.
Therefore, as AE is to EB, so is CG to GD. - Ax. 13.5.

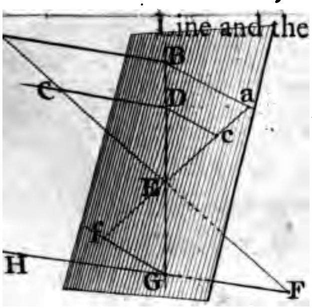
COR. Hence it is manifest, that any Number of Right Lines, proceeding from one Point, are cut proportionally (from their common Point of Section) by parallel Planes.

For, since, DF : DA : : DG : DC; 2. 6. & 6. 5. consequently, any other Right Line, DH, is cut in the same Ratio, at H and I.

THE-

THEOREM XI.

If two Right Lines are perpendicular to a Plane, (whether they are on the ſame Side, or on contrary Sides,) and are both cut by another Right Line, which alſo cuts the Plane; that Line will cut the Plane, ſomewhere in a Right Line paſſing through the Points where the Perpendiculars cut the Plane; and, it will be cut in the ſame Ratio, by the Plane, and the Perpendiculars, as the Perpendiculars, by that Line; and the Right Line, joining or paſſing through the Points where the Perpendiculars cut the Plane, will alſo be cut proportionally, by that Line and the Perpendiculars.



Firſt; let the two Lines AB and CD, (on the ſame Side) be perpendicular to the Plane a B f; cuting it in the Points B and D.

From any Point A, in AB, draw a Right Line AC, cuting CD, in C, and produce it, till it cuts the Plane at E.

Then, a Right Line, BD, being produced, will paſs through the ſame Point, E; and, they will, both, be cut, in the Points A, C, E, and B, D, E, in the Ratio of the two Perpendiculars; viz. as AB to CD.

DEM. For, becauſe AB and CD are both perpendicular to the ſame Plane, they are parallel. - - - Th. 3.
wherefore, AC and BD are in the ſame Plane. - Ax. 3.
And, becauſe the Points B and D are in the Plane a B f; the Right Line BD, is in that Plane - - Ax. 1.
(for it is the common Section of the two Planes)
Conſ. every Line (as AC) not parallel to BD, in the Plane AEB, will cut the other Plane, in the Interſection of both.
Th. AC produced, will cut the Plane a B f, in BD produced.

2nd. Becauſe CD is parallel to AB, the Sides EA, EB, of the Triangle AEB, are cut proportionally. - - 2. 6.
wh. as EC:CA::ED:DB; conſ. EC:EA::ED:EB -6.5.
i. e. EC:EC+CA::ED:ED+DB; viz. as CD:AB -4.6.

Again. Let FG (on the other Side of the Plane) be perpendicular to the ſame Plane (a B f) with AB; they are parallel (3.) which is evident, producing either; as FG, to H

Then, if AF be drawn, cuting AB and FG, it will alſo cut the Plane a B f in the Line BG, joining the two Points, B and G, in which the Perpendiculars cut the Plane.

And, they alſo, mutually, cut each other, in the proportion of AB to FG.

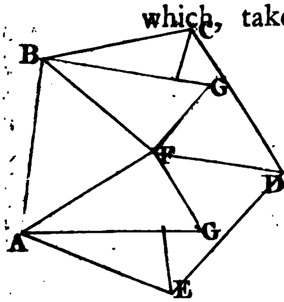
DEM. Becauſe A B is parallel to F G, a Plane may paſs through them both. - .. - - - Ax. 2.
conſequently, AF and BG are both in the ſame Plane with AB and FG. - - - - Ax. 3 and 4.
Wherefore, in the Triangles, EAB, EFG, the Angles at E being vertical, are equal (2. 1.) the Angles at B and G are right (2. 7.) therefore equal - - Ax. 10. 1.
the Angles at A & F, are conſ. equal; and alſo, by Th. 4. 1.
Therefore, thoſe Triangles are ſimilar; - - 4. 6.
and therefore, as AB : FG : : AE : EF, and BE to EG.

COR. Hence it is manifeſt, that if a Right Line, a B, be drawn, at pleaſure, through the Point B, in the Plane a B f; and through G, if f G be drawn parallel to a B; a B and f G being made equal to AB and FG, reſpectively, or in the Ratio of AB to FG; then, a Right Line, drawn through a and f, will cut BG in the ſame Point E, as before.

For, becauſe a B is parallel to f G, the Triangles E a B, E f G, are ſimilar; or, becauſe a B is parallel to f G, a f and BG are cut proportionally, in E. - - - - Caſe 3. 2. 6.
And, becauſe the Triangles are ſimilar, a E : E f : : BE : EG, and, as a B : f G. - - - - - - - 4. 6.
But, a B:f G::AB:FG; conſ. a E:E f, and BE:EG,::AB:FG; and therefore, BG is cut in the ſame Point (E) as before.

THEOREM XII. 21 Euclid.

Every solid Angle is contained under Plane Angles; which, taken together, are less than four Right Angles.



Let ABCDE be a Section of the Planes AFB, BFC, &c. forming a solid Angle, at F; i. e. let ABCDE be the Base of a Pyramid, whose Vertex is F; supposed to be elevated out of the Plane of the Base.

Dem. Now, all the Angles AFB, BFC, &c. being in a Plane, are equal to four Right Angles. - - Cor. 2. 2. 1. But, if F be supposed elevated out of that Plane, equal to FG, and perpendicular to AF, BF, &c. and AG, or BG, drawn, AG is longer than AF - - - 12. 1.

For the same reason, all the Right Lines, from each Angle, A, B, &c. of the Base, to the Vertex, at F, elevated perpendicularly, at F (eq. FG) are longer than BF, CF, &c.

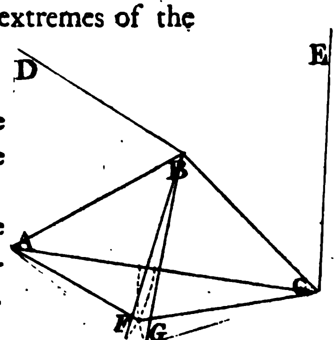
But, if two Sides of a Triangle be, respectively, greater than two Sides of another Triangle, and the remaining Sides are equal, they shall contain a less Angle - 14. 1.

Therefore, the Angles of the Triangles AFB, BFC, &c. being considered as raised out of the Plane ABCDE, are less, respectively, than those Angles in the Plane; which, being equal to four Right Angles, consequently, the Angles elevated out of the Plane, are less than four Right Angles.

If the Triangles AGF, BGF, be turned up, on AF & BF, till FG, in one, coincide with FG in the other; and from the Point G, so elevated, (FG being perpend. to the Plane ABD) if Right Lines are drawn to all the other Angles, C, D, and E; each two will contain a less Angle than those on the Plane, to which they are opposite, (as AGB than AFB) and are, together, less than four Right Angles.

THEOREM XIII. 20 & 22 Euclid.

If any two, of three given Plane Angles, be greater than the third, and the Lines containing the Angles all equal; and if thofe three Angles are, together, lefs than four Right Angles, they will form a folid Angle; and three Right Lines, joining the extremes of the equal Sides, will form a Triangle.



Let ABC, ABD, and EBC be three Plane Angles; of which, any two are greater than the remaining Angle; and, the Sides, AB, BC, BD, and BE are all equal; alfo, the three Angles, together, are lefs than four Right Angles.

Then, a folid Angle may be formed of thofe three Angles, ABC, &c. alfo, of the three Lines, AC, AD, and EC, a Triangle may be formed.

DEM. Turn over the Triangles AFB and BGC (equal ABD and EBC) till the two Sides BF and BG coincide.

Becaufe they are lefs than four Right Angles (the deficiency being the Angle FBG) it is manifeft they will form a folid Angle, at B, their common Vertex; the two leffer Angles, ABF, CBG, being together greater than ABC.

For, it is evident, if they were lefs, as ABF, CBG, the Sides BF and BG would not meet each other; and conf. with ABC, they cannot form a folid Angle, at B.

Therefore, a folid Angle formed of three Plane Angles, any two are, together, greater than the third. Q. E. D.

Or, a solid Angle formed of any number of Plane Angles, any one must be less than all the others, added together.

Nothwithstanding, if BF and BG be equal to AB & BC respectively, although the Angles (ABF, CBG) are, together, less than ABC, AF & CG, together, are greater than AC; and consequently, a Triangle may be formed of the three Lines AC, AF and CG; and therefore, of AC, AD, and EC; which are greater than AF and AG - 13. 1.

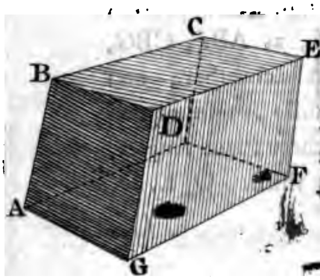
N. B. The last part of this Theorem, which (according to Euclid) seems to imply, that, unless two of the Plane Angles are greater than the third, a Triangle cannot be formed of the three Lines, joining the extremes of equal Sides; whereas, the contrary is manifest. It appears however, to me, of so little consequence, as not worthy of being made a distinct Theorem, of itself.

N. B. 2. To make a Solid Angle of three given Plane Angles, (the 23rd Prop. of Euclid) is to place them so, together, that the Vertices, of all the three, meet in one Point; and, the Sides, which contain the plane Angles, coincide, two and two in one Line, as in the Figure. (see Defin. 6.)

With this Problem, Euclid's Commentators take up more than four Pages; but, for what purpose I can't devise.

THEOREM XIV. 24 Euclid.

If a Solid be contained within six Planes, of which two and two are parallel; the opposite Planes are Parallelograms, equal and similar.



Let ABEG be a Parallelopiped.

The opposite Planes ABCD, DEFG; BCED, and ADFG; also, ABDG, and DCEF, are, respectively, equal and similar.

DEM.

DEM. The Planes, AC, GE, being parallel, and both cut by the Planes BG, and CF, which are alſo parallel; their Sections, AB and DG, CD and EF, are parallel amongſt themſelves. - - - - - - Th. 8.

But, they are alſo cut by the parallel Planes, BE and AF; their Sections, AD, BC, DE, and FG, are conſ. parallel.

Therefore, the Figures formed by thoſe Sections are Parallelogms. ſeeing their oppoſite Sides are parallel-Def. 33

2nd, Becauſe AB is parallel to GD, and AD to GF, and are in parallel Planes, the Angle BAD is equal to DGF; wh. the oppoſite Angles, BCD and DEF, are equal-15.1. But AB = CD, and alſo to DG; and DG = EF-ſame. conſ. they are all equal amongſt themſelves. - Ax. 3. 1. Alſo, AD, BC, DE, and GF are equal amongſt themſelves; therefore, the Parallelogram ABCD = DEFG, and ſimilar; alſo, BE = AF, and BG to CF, and reſpectively ſimilar.

THEOREM XV. 25 Euclid.

If a Parallelopiped be cut in two Parts, by a Plane, parallel to any two of its Planes; the two Solids will have that Proportion to each other, as their Baſes (on which they are ſuppoſed to ſtand) i. e. as the Parts of any Plane cut by the other; or, as the Segments of the Sides.

This Theorem is manifeſtly an Axiom.

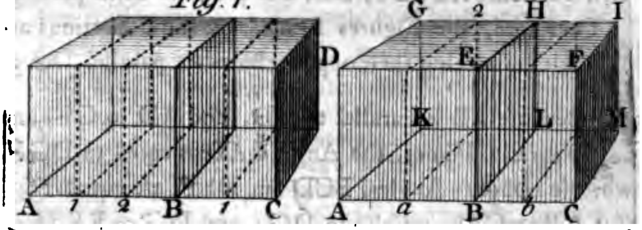
For, ſince the Sections made by parallel Planes are equal, every where, which is manifeſt, ſeeing the oppoſite Planes are equal and ſimilar; conſequently, if each Part be again cut, into Parts, by parallel Sections, which are equal amongſt themſelves (as in Fig. 1.) the Demonſtration follows from the 9th Axiom of the 5th Book.

Or,

Or, being incommensurable, they may be divided into an equal Number of Parts, (Fig. 2.) and demonstrated as Theorem 1st.

If the Base (AKMC) or Sides (AC, or DF) are bisected, the Parallelopiped is bisected; and consequently, in whatever Ratio one is divided, (by a Plane BEHL) the other is the same.

Fig. 1.



Dem. Let the Base AM, or any other Plane, AF, be cut by a Right Line, parallel to its Sides, AK or AD; and imagine a Plane (BH) to pass through BE or BK, parallel to AG; then, AE : EC, or AL : LC :: AB : BC. - - Th.1.6.
And, seeing the opposite Planes, DGIF and GM, are equal and similar, to AKMC, and AF, respectively; and are all cut by the Plane BEHL, parallel to AG and CI; consequently, the Solid AGHB : BHIC :: AE : BF, or, AL : BM, i. e. as AB : BC.

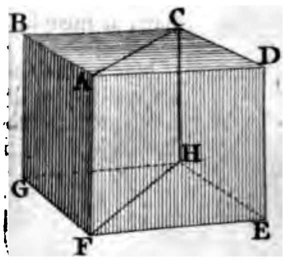
N. B. The two following Propositions, 26th and 27th of Euclid, are Problems; which, I do not conceive to be at all necessary, or elementary.

To make Solids, is rather a mechanical Process than geometrical; 'tis sufficient, for the purpose of Demonstration, to suppose that one solid Angle, or one Solid is equal to another; for, an attempt to make them really so, on a Plane, is inconsistent; unless it be to delineate them, perspectively so.

THE-

THEOREM XVI. 28 Euclid.

If a Parallelopiped be cut by a Plane, paffing through the Diagonals of two oppofite Planes, it will be cut in two equal Parts.



Let A B D F be a Parallelopiped, and AC; FH, Diagonals of the oppofite Planes, A B C D and E F G H.

Then, becaufe A F and C H are both parallel to B G, they are parallel between themfelves (Th. 4.) wherefore, AF and CH are in the fame Plane - Ax. 3.

I fay, the Parallelopiped, A B D F, is bifected by the Plane A C H F.

Dem. For, the Triangle ABC=ADC; FGH=FEH - 15. 1. And the Parallelogram ABCD is equal to EFGH - Th. 14. (wh. the Triangles are all equal amongft themfelves.) But, the Parallelogram BF=CE, & AE=BH; - Th. 14. alfo, the Triangles ABC, and F G H; A D C, and F E H are equal to one another - - - - - above. wherefore, the Prifm F B C is equal to CEF - Ax. 5. (For, they are contained within the fame number of Planes which are equal, and fimilar to each other, refpectively) Therefore, the Plane, A C H F, paffing through oppofite Diagonals, (AC and FH) bifects the Parallelopiped, ABDF.

Otherwife.

By the parallel motion of the Parallelogram A B C D along the Right Lines A F, D E, &c. the Parallelopiped is fuppofed to be generated. (fee Note, Def. 8.)

Wherefore, fince the Triangles, ABC, ACD, are equal, and the Prifm, FBC, CEF, are defcribed in equal Time, that is, by the fame motion of both together, confequently, they are equal.

4

The Definition of a Prifm, as it is given by moft Authors, is fomewhat obfcure and unfatisfactory; and, we are left to imagine, that any one of the Planes may be its Bafe; we may as reafonably conclude, that any Plane of a Pyramid may be its Bafe, but it is never fuppofed fo, unlefs they are all Triangles; nor in a Prifm, except they are all Parallelograms.

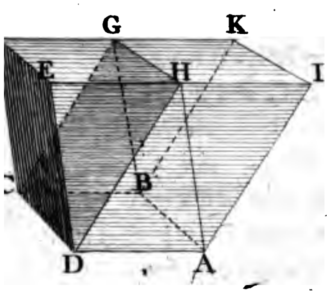
For, to fuppofe the two equal, fimilar, and parallel Planes, in a Prifm, Triangles or Poligons, and any other Plane to be its Bafe, is inconfiftent.

Hence, the 40th Propofition of Euclid, which, in reality, is but a Corollary to this Theorem, is an abfurd Propofition.

Becaufe, I do not conceive, that, unlefs a Prifm be alfo a Parallelopiped, its Bafe can be a Parallelogram.

THEOREM XVII. 29, 30, & 31 Euclid.

Parallelopipeds having the fame Bafe,* or equal Bafes, and the fame Altitude, are equal to one another.



Firft; let the two Parallelopipeds, AHFC and AIFC, have the fame Bafe, ABCD; and, let their oppofite Faces have a common Side, GH.

I fay, the Parallelopiped A I C is equal to AKGC.

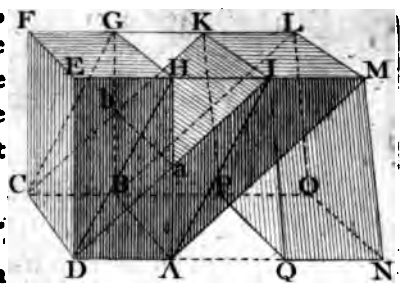
---

* By the Bafe, of a Solid, literally, is underftood the Plane or Face, on which it is fuppofed to reft; but, it is not always confidered fo, in Geometry,

When any two Faces of Parallelopipeds, are in one Plane, or in parallel Planes; whether they coincide with each other, entirely, or are apart, and diftinct from each other (no regard being had to the pofition of the Plane they are in) the Top, in fuch Cafe, is fometimes called the Bafe.

N. B. Parallelopipeds, having the fame Altitude, are fuppofed to be contained between the fame parallel Planes; i. e. two of their oppofite Planes, in each Solid, are in the fame Plane; or, that there is the fame perpendicular diftance between them.

DEM. The Parallelopiped AHFC, being cut by a Plane, CGHD, through the oppoſite Diagonals, DH and CG, is cut into two equal Parts CAG = DFH. - Th.16.
And, for the ſame reaſon, the Plane AHGB biſects the Parallelopiped AIGC; in AIKG equal ADCG.
But, ADCG + DFH = ADCG + AIKG - Ax.6.1.
Therefore, the Parallelopiped AHFC = AIGC. Q.E.D.



Secondly. Let the Parallelopipeds, AHFC and AMKC, have the ſame Baſe, ABCD; their oppoſite Faces, EFGH and IKLM, are in the ſame Plane, and have not a common Side.

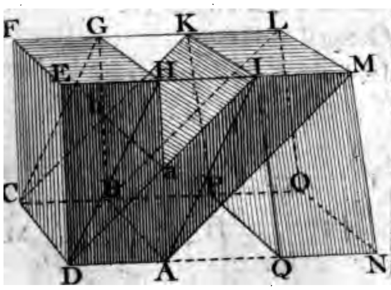
AMKC is equal to AHFC.

Let the ſpace, GHIK, between the upper Faces, be equal to the Baſe ABCD; conſ. equal to EFGH, equal IKLM-Th.14.

DEM. The Parallelopiped AHFC is equal to AIGC; -proved. and, for the ſame reaſon, AIGC is equal to AMKC. Th. the Parallelopiped AMKC is equal to AHFC-Ax.3.1.

Again. Suppoſe the ſpace, GHIK, be either greater or leſs than EFGH.
The Triangles, AMH, DIE, are congruous; - 7. 1.
(for AM is equal to DI, MH = IE, and AH = DE; by 15.1.)
And the Sides, AB, DC, EF, GH, IK, & LM, are equal -14.
the Par. AG = DF; ABLM = IKCD; & EFKI = MLGH;
conſ. the Solid AGLM is equal to DFKI. - Ax. 5.
Now, if the Solid, aIKGb (common to both) be taken away,
AMKb is equal to DaGC. - - Ax.7.1.
And, if the common Solid AabCD be added, to both;
the Parallelopiped, AMKC, is equal to AHFC - Ax.6.1.

The ſecond Part, viz. when they have equal Baſes only, and have not one common to both, is readily proved.



Let the Baſes, ABCD, of the Parallelopiped AHFC, be equal to Baſe NOPQ, of the Parallelopiped NMKP (and ſimilar)

Then, the Parallelopiped, AHFC, is equal to NMKP.

Let them be ſo ſituated, that the Planes AE, and NI, are in the ſame Plane, DEMN; and draw AM, DI, BL, and CK.

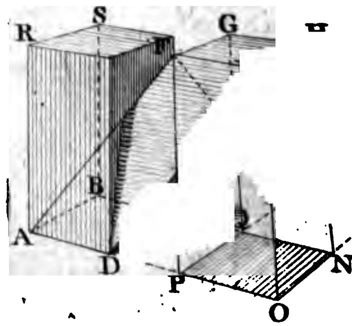
Dem. Now, becauſe MI is parallel to AD, and LK to BC, and alſo equal, AMID and BLKC are Parallelgms - C.15.1. And, for the ſame reaſon, AMLB and DIKC are Parallelograms; and they are, alſo, parallel and ſimilar, reſpectively; wherefore, AMKC is a Parallelopiped. - - Def. 9. (For, AM is parallel to DI, and ML to IK; wherefore, the Angle AML = DIK, &c. - Th. 5.)

But, the Parallelopiped AHFC has the ſame Baſe with AMKC; and they are contained within parallel Planes; wherefore, they are equal Parallelopipeds. - - proved. Alſo, the Parallelopiped NMKP, has the ſame Baſe with AMKC (i.e. the Plane or Face IKLM is common to both.) conſequently, they are equal Parallelopipeds. - ſame. But, the Parallelopiped AMKC is, alſo, equal to AHFC. Therefore, the Par. NMKP, is equal to AHFC - Ax. 3.1.

After the Proofs already given, it may, by ſome, be thought unneceſſary to add more; but, there are other Caſes, in which there is abundantly more room to diſpute the Aſſertion, than in the preceding; and, as this Theorem is very eſſential, it cannot be too much enforced.

Case II.

CASE II. When no more than two Faces of the Parallelopipeds (i. e. two in each) are in the same parallel Planes.



Let AGHD and FHIL be Parallelopipeds; whose Bases, ABCD and IKLM, are equal, and in the same Plane; and the Top, EFGH, common to both; but have no other Plane common.

I say, that the Parallelopiped AGHD, is equal to FHIL.

Produce AD and BC, IM and KL, (being in the same Plane) they will cut each other, in NOPQ, which is a Parallelogram; equal and similar to ABCD, and IKLM.

DEM. Because EFGH is common to both Parallelopipeds, and because the opposite Planes are equal and similar, AC is similar to LI; and, being similarly posited, IK, LM, are parallel to AD and BC; and IM, KL, to AB and DC; therefore, NOPQ, being between the same Parallels, is consequently equal and similar to AC, & LI - 4. & 18. 1. and consequently, to EFGH; and also similarly situated.

Join EO, FP, GQ and HN.

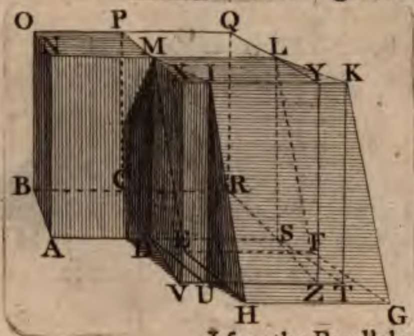
Then, PFHN is a Parallelopiped; which, being equal to each of the other, AGHD and FHIL (part 1st) they are consequently equal between themselves - Ax. 3. 1.

Also; if the Parallelopipeds, ASFD and FHIL, have their Bases equal; and have no Face common; being between the same parallel Planes, AOI and RGE, they are equal Parallelopipeds.

This is evident, from the foregoing; each being equal to PFHN.

 CASE III.

CASE III. When the Parallelopipeds have no Face common, nor similar Figures.



Let the Parallelopipeds AOMD and EMKG, be between the same parallel Planes, ARH and NQI; consequently, they have equal Altitude; and, let the Base, ABCD, be equal to the Base, EFGH, of the other, which are not similar Figures.

I say, the Parallelopiped, EMKG, is equal to AOMD.

DEM. Let the Side NM, of one, and ML of the other, be so placed, at the common Angle M, that NML is a Right Line.

Compleat the Parallelopiped, DMKT; having a common Side, DM, with AOMD.

The Parallelopiped DMKT is equal to EMKG - Case 2nd.
(For they have the same Altitude, and IMLK is common)

Produce KI and TU, PM & CD, meeting at X and V. Draw LY and SZ, parallel to MX and DV, cuting IK and TU, at Y and Z. Join VX and YZ.

Then the Parallelopiped VMYZ = UMKT - 1st.
conf. the Parallelopiped, VMYZ = HMKG - Ax. 3. 1.

Produce OP and YL, BC and ZS, meeting at Q and R; and, joining QR, there is formed a Parallelopiped DPQS; equiangular to both the Parallelopipeds, AOMD and VMYZ; conf. they are equiangular amongst themselves.

Now, the Parallelogram IMLK = NOPM - Hyp.
XMLY (equal IMLK 18. 1.) is equal to NOPM - Ax. 3. 1.
consequently, (the Angle PMN being equal XML - 2. 1.)
as NM : ML :: XM : MP. - - - 8. 6.

4 But

But, the Parallelopiped AOMD : DPLS :: NM : ML } 15.
And, the Parallelopiped VMYZ : DPLS :: XM : MP; }
i. e. as NM : ML. - - - - - above.
Wherefore, the two Parallelopipeds, A O M D and V M Y Z, have an equal Ratio to D P L S;
and therefore they are equal. - - - Ax. 4. 5.
But, the Parallelopiped HMKG = VMYZ - above.
Therefore, H M K G, is equal to A O M D.

Corollary I. The 33rd Propofition of Euclid.

Parallelopipeds having equal Altitudes, have that Ratio to each other as their Bafes.

For, fince by this Theorem, all Parallelopipeds, having equal Bafes and equal Altitudes are equal; and, by the 15th, if a Parallelopiped be cut in two, by a Plane, parallel to its Planes, the two Parts have that proportion to each other as their Bafes; confequently, all Parallelopipeds, having equal Altitudes, are to each other as their Bafes.

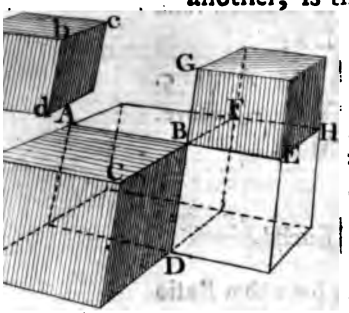
Cor. II. Parallelopipeds, having equal Bafes, have that Ratio to each other as their Altitudes.

For, having equal Bafes and equal Altitudes, they are equal; confequently, if the Altitude of one, be half, or a third part, &c. of another; or, in whatever Ratio the Altitude of one Parallelopiped is to the other, whofe Bafes are equal; they are in that Proportion, to each other, as their Altitudes.

THE-

THEOREM XVIII. 33 Euclid.

The Proportion of similar Parallelopipeds, to one another, is the triplicate Ratio of their correspond-ing Sides.



Let ADK and a c d be similar Parallelo-pipeds; in which, let AB, a b, BC, b c, and B D, b d be corresponding Sides, in each Solid.

Then the Proportion, which the Solid ADK has to a c d, is the triplicate Ratio of AB to a b, BC to b c, or BD to b d.

In the Solid ADK, produce the Sides, AB, CB, & DB. Make BE equal to a b, BF equal b c, and BG equal b d; draw EH and FH, respectively, parallel to BF and BE; and compleat the Parallelopipeds GH, DH, and AFD ; equiangular to A D K.

Dem. Because B H is a Parallelogram (by Construction) and the Angle EBF = ABC, (2.1.) (EBG = ABD, and FBG = CBD) also BE = a b, and BF to b c; BH is equal and also similar to a c; for, a c is similar to AC; by Hyp.

And, for the same reason, GE, is equal and similar to a d (which is similar to AD) also, F G is equal and similar to c d (similar to CD) consequently, the Parallel-opiped, GH, is equal and similar to a c d (similar to AKD)

Now, AB : a b : : BC : b c, and as BD is to b d - Def. 11.7. and, the Parallelopiped G H is equal and similar to a c d; BE = ab, &c. wh. as AB : BE : : CB : BF, and DB : BG. But, the Parallelopipeds, A F D and D F H, have equal Altitude; with A D K.

wh. they have that Proportion to each other as their Bases;
i. e. the Solid ADK : AFD : : Par. AC : AF; i. e. as CB to BF } Cor. 1.
and, the Solid AFD : DFH : : Par. AF : FE; i. e. as AB to BE } of 17.
also, - - DFH : GEH : : Par. DE : EG; i. e. as DB to BG } & 1. 6.

wherefore, the Parallelopiped ADK : AFD : DFH : GEH ∺ in continual Ratio. (for the Ratio of CB to BF, AB to BE, and of DB to BG, is the fame, by Hypothefis.)
Therefore, the Ratio of ADK to a c d (equal GH) is triplicate of ADK to AFD; i. e. of CB to BF (eq. b c) of AB to BE (eq. a b) or, of DB to BG (eq. b d)-Def. 12. 5.

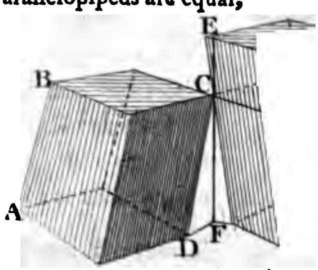
THEOREM XIX. 34 Euclid.

The Bafes and Altitudes, of equal Parallelopipeds, are reciprocally proportional.

This Theorem will admit of as many Cafes as the 17th; but one general Cafe, being clearly demonftrated, will be fufficient.

Let ABD and FEH be equal Parallelopipeds.

If the Bafe of one be equal to the Bafe of the other, their Altitudes are alfo equal (Th. 17.) becaufe the Parallelopipeds are equal, by Hypothefis. Let them be unequal.



Then as the Bafe AD is to FG, fo is E F to C F, their Altitudes.

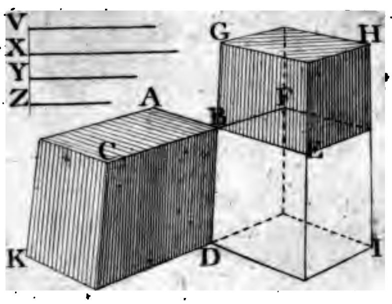
Let them be fo placed together, that the Angle C, of the one, touches the Side E F, of the other.

Produce the Plane, BC, to H s parallel to FG; and, through C, let E F be drawn, perpendicular to their Bafes, AD, FG (which are fuppofed to be in the fame Plane) EF and CF are the Altitudes of the Parallelopipeds.

DEM. Then, ABD : FCG : : AD : FG, their Bafes - Cor. 1. 17
And, - FEG : FCG : : EF : CF, their Altitudes-Cor. 2. 17
But, - ABD is equal to FEG; by Hypothefis.
Therefore, - ABD : FCG, i. e. AD : FG : : EF : CF
i. e. as the Bafe AD, of one, is to FG of the other;
fo is EF the Altitude, of the laft, to CF of the firft.

THEOREM XX.

Parallelopipeds, whofe folid Angles are equal, have that Proportion, to one another, which is compounded of the Ratio of their Sides.



Let ADK and BIH be equiangular Parallelopipeds; and let them be fo placed together, at the equal Angles B, that, the Side AB of the one, is in the fame Right Line with BE, of the other, and CB with BF; confequently, the two Sides BD and BG are in one Right Line.

(It is not material which Plane, EF, GE, or GF, is made the Bafe.)

On EF, and BD, compleat the Parallelopiped DFI.

Take any Right Line, V, at pleafure; and make X to, V as BE to AB; and as BF is to CB, fo make Y to X; alfo, as BG is to BD; make Z to Y. - (Prob. 32.)

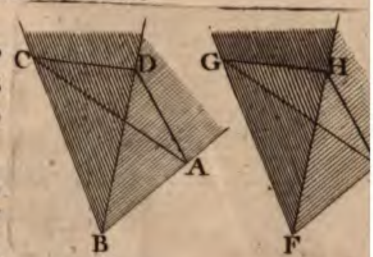
Then, as V is to Z, fo is the Parallelopiped, ADK to BH.

DEM. The Solids, ADK and DFI, have equal Altitudes; wherefore, they are to each other as AC to EF; -C.1.17. that is, as the Ratio which is compounded of AB to BE, and CB to BF; i. e. as V to Y. - - Th.11.6.
And, the Solid DFI is to BH, as the Par. DE is to EG - 17.
that is, as DB is to BG, i. e. as Y to Z. - 11. 6.
But, ADK : DFI : : V : Y, and DFI BH : : Y : Z -Con.
Therefore, the Solid ADK : BH : : V : Z - - Th. 9. 5.
i. e. in the compounded Ratio of AB to BE, CB to BF, and DB to BG.

THEO-

THEOREM XXI.

In ſolid Angles contained by three Plane Angles, two and two of which (i. e. one in each) being equal, reſpectively; thoſe Planes, which contain the equal Angles, have equal Inclination to one another.



Let B and F be two ſolid Angles, each contained by three Plane Angles; viz. ABC equal to EFG, ABD equal to EFH, and CBD equal to GFH.

I ſay, the inclination of ABD, or CBD, to ABC, is equal to the inclination of EFH, or GFH, to EFG; alſo, the inclination of ABD to CBD is the ſame, as EFH to GFH.

In AB take any Point, A, at pleaſure, and take EF equal AB; and, in the Planes ABC, ABD; EFG, EFH, draw AC and AD, EG and EH, at right Angles with AB and EF, reſpectively, and join the Points, C and D, G and H, by Right Lines.

DEM. The Angle ABC = EFG (Hyp.) and BAC = FEG - Con. and AB = EF; wh. the Triangles ACB, EGF are congruous; AC is equal to EG, and BC is equal to FG. - Th. 11. 1.
Alſo, the Angle ABD = EFH (Hyp.) & BAD = FEH - Con. wherefore (ſeeing AB = EF) BD = FH, and AD to EH - ſame
And, becauſe BC, BD, are reſpectively equal to FG, FH, and the Angle CBD is equal to GFH; CD = GH - 8. 1
Laſtly, becauſe the Triangle ACD is equilateral to EGH. they are alſo equiangular; and, the Angle CAD = GEH - 7. 1
But, thoſe Angles are the Inclination of the Planes, ABD to ABC, and EFH to EFG. - - Def. 5.
Therefore they have equal Inclination. Q. E. D.

After the ſame manner, it may be proved that any other two have equal Inclination; therefore, &c.

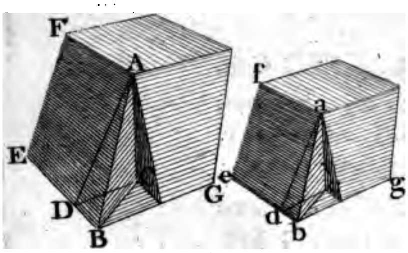
COR. Solid Angles, contained by three Plane Angles, each, of which, two and two (one in each) are equal, respectively, are equal to one another.

Becaufe the Planes have equal Inclination to each other, refpectively (by the Theorem) Therefore the Angles are equal. *

THEOREM XXII.

If from equal folid Angles, in equiangular Parallelopipeds (which are not right angled) Perpendiculars be drawn, fimilarly, i. e. to equiangular Planes; the Perpendiculars (i. e. their Altitudes) will be proportional to the Sides, which are inclined to the Planes on which the Perpendiculars fall.

This Theorem, though very different, apparently, is the fame, in Subftance, as the 35th of Euclid.



Let B F G, b f g be equiangular Parallelopipeds, and let the Angle A be equal to a; confequently, B is equal to b.

From the equal Angles A, a, let the Perpendiculars AC, a c, be drawn to equiangular Planes (their Bafes) containing the equal Angles E B G, e b g; cuting them at C and c.

I fay, that, as AB is to a b, fo is AC to a c.

Through C and c, draw CD and c d, perpendicular to BE and b e, refpectively; and join AD and ad.

* Profeffor Simfon fays (Prop. B.) "and alike fituated" making that a Condition of their equality; whereas, it is not; for, it is impoffible, that three Plane Angles, equal refpectively to three other Plane Angles (containing a folid Angle) can be fo placed, in forming a folid Angle, which will not be equal to the other.

For, they muft neceffarily have the fame Inclination each to the other, refpectively, however fituated.

DEM. Becaufe AC is perpendicular to the Plane EG - Con.
the Plane, CAD, paffing through AC, is perp. to EG-Th.9
and, for the fame reafon, c a d is perpendicular to e g.
But, CD is perpendicular to BE; and c d to b e; - Con.
wh. BE is perpendicular to the Plane CAD, and b e to cad;
conf. AD is perpendicular to BE, and a d to b e. - Th.2.
Then, in the Triangles BAD, b a d, the Angles ADB,
a d b are right, therefore equal - - - Ax. 10.1.
and the Angle ABD = a b d, (Hyp) conf. BAD = b a d - 10 1
and therefore, as AB : AD :: a b : a d - - Th. 4.6.
But, the Angle ACD = a c d (becaufe, AC is perpendicular to the Plane EG, and a c to e g, ACD, a c d are Right Angles) and, the Angle ADC = a d c, - 21
for, A D C is the Inclination of the Plane AE to EG;
and, a d c is the Inclination of the Plane a e to e g.-Def. 5.
wherefore, the Triangles DAC, d a c are fimilar;
and, therefore as AC : AD :: a c : a d. - - Th. 4.6.
But, AD:AB::ad:ab (above) Th. AC:AB::ac:ab - 9 5.
and, by alternation, AC : a c :: AB :: a b. Q. E. D.

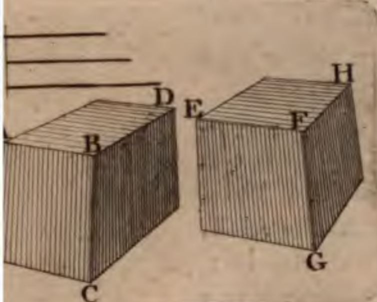
THEOREM XXIII. 36 Euclid.

If three Right Lines are Proportionals; a Parallelopiped defcribed, or conftructed under all three, for its Sides, is equal to an equiangular Parallelopiped, whofe Sides are each equal to the Mean.

If the Parallelopipeds be right angled, that which is conftructed under the Mean is a Cube.

Let the three Right Lines X, Y, and Z, be Proportionals; as X is to Y, fo let Y be to Z.

 And,



And, let ACD be a Parallelopiped, whose Side AB is equal to X, BC equal to Y, and BD equal to Z.*

Alſo, let EGH be a Parallelopiped equiangular to ACD.

Then, if the Sides EF, FG, and FH, are each equal to Y; *
the Parallelopiped ACD is equal to EGH.

DEM. Becauſe, X:Y::Y:Z, the Rectangle under X and Z, is equal to the Square of Y, i.e. $X \times Z = Y\square$ - Cor. 9. 6.
But, Parallelograms, having equal Angles, are in the ſame Ratio as Rectangles, whoſe Sides are equal to the Sides of the Parallelograms, reſpectively, - - C. 2. 11. 6.
wherefore, the Parallelogram AD is equal to EH.
But, $BC = FG$; and conſ. their Altitudes are equal - 22
Therefore, the Parallelopiped ACD is equal to EGH.
For they have equal Baſes, and equal Altitudes.

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* In thoſe Caſes, the Figures being but repreſentations of Solids, on a Plane; if the Lines, BD, and FH, were made equal to Z, and Y, reſpectively, they could not poſſibly have that appearance, but they would appear much greater; becauſe they are ſuppoſed to recede, from the Planes of the Faces; AC and EG.

Therefore, they are ſuppoſed to be equal, only; which is the ſame thing, and holds equally true, in the Demonſtration. If one be, the other is a neceſſary Conſequence.

Note. As the Ratio, between two ſimilar Plane Figures, is diſcovered by a third Proportional, ſo between Solids, a fourth determines it.

Alſo, to deſcribe plane Figures, in any given Ratio, a Mean Proportional is found; ſo in reſpect of Solids, two Means are requiſite (ſee Prob. 33. or P. 332. B. 6.)

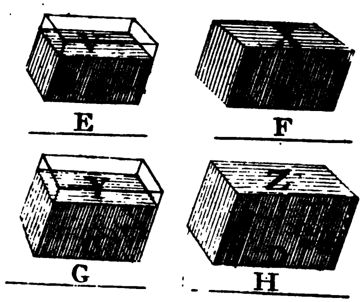
For, ſince the Ratio between ſimilar Solids is triplicate of their correſponding Sides; conſequently, ſince AB : DH : BF : AD ÷; and, if the Ratio of two ſimilar Solids be as AB to AD, AB and DH are in the Ratio of their correſponding Sides.

Hence, ſimilar Solids may be conſtructed in any Ratio to one another.

THEO-

THEOREM XXIV. 37 Euclid.

If four Right Lines are Proportionals; ſimilar Parallelopipeds, and ſimilarly conſtructed on each Line, are alſo Proportionals.



Let A, B, C, and D be Proportionals; as A is to B, ſo let C be to D.

And let the Parallelopipeds V, X, Y, and Z, be ſimilarly conſtructed, on A, B, C, and D.

I ſay, that, as V is to X, ſo is Y to Z.

DEM. Let E and F be ſo taken, as A:B:E:F ÷ } (Prob.31)
And, let G and H be taken, as C:D:G:H ÷ }
Then, the Parallelopiped V : X : : A : F } - Th. 18.
And the Parallelopiped Y : Z : : C : H }
But, as A:B::C:D; wherefore, as B:E::D:G; and, as E:F::G:H; (Ax.13.5) conſ. as A:F::C:H - Th.9.5.
Therefore, as the Parallelopiped V:X::Y:Z. Q. E. D.

It is not neceſſary that the Parallelopipeds Y and Z ſhould be ſimilar to V and X; for, if V be ſimilar to X, and Y to Z, the Demonſtration is evidently the ſame.

Or, if V be not ſimilar to X, nor Y to Z;
but, let V be ſimilar to Y, and X to Z; it ſtill holds true.

For, ſince A:B::C:D, ſo A:C::B:D - - Th.4.5.
conſequently, ſimilar Parallelopipeds, conſtructed on A and C, and others, not ſimilar to them, on B and D, will ſtill be Proportionals.

For, ſince the Parallelopipeds, on A and C, B and D, are, as V:X::Y:Z, ſimilar; ſo V:Y::X:Z, not ſimilar.
The ſame holds true, in ſimilar Solids of any kind.

E L E-

ELEMENTS
OF
GEOMETRY.

BOOK VIII. The XII. of Euclid.

AS the laſt Book, or Section, treats only of Plane Solids, and determines their Proportions, their Properties and Relation to each other; ſo, in this, the Doctrine of Solids, bounded by regular curved Surfaces, as the Cylinder, Cone, and Sphere, are conſidered; the Solution of which is truely admirable.

It muſt appear, to all who conſider it, before they have gone through this Book (or ſome other on the Subject), a matter not merely of difficulty, but of abſolute impoſſibility, to aſcertain the true Meaſure or Area of the Surface, much leſs of the ſolid Contents of a Sphere; to reduce ſuch a Solid to right angled and cubical Meaſure, with any degree of certainty, ſeems to be beyond the power of Art, or the reach of finite knowledge.

And yet, nothing more is requiſite thereto, than the true meaſure of its Diameter, which may be acquired; at leaſt of any Sphere formed or conſtructed by Man. In reſpect of the Earth itſelf, its Diameter can only be obtained by its Circumference. It is poſſible, by various means, to come ſomewhat near the truth, but it is beyond human abilities to aſcertain it, with any degree of abſolute certainty.

How much leſs, then, are we able to come at the true Diſtances, by which, the magnitudes of diſtant Planets are reduced to the ſame Scale of Proportion; viz. by the known

meaſure

meaſure of the Earth. For, if the one cannot be had, with certainty, how much more incorrect muſt be the other, having no other Data to build on ?

Although the many ingenious contrivances, by Inſtruments curiouſly conſtructed for that purpoſe, are worthy of the greateſt praiſe, and redound highly to the Fame of their ſeveral Inventors ; and notwithſtanding all other means, by Tranſits, &c. may ſtill approch nearer to the truth, or, at leaſt, evince and prove the imperfections of all human Inventions, for the purpoſe ; we muſt (or may as well) remain ſatisfied, that we have acquired ſo much, as to anſwer various great and notable purpoſes, in Trigonometry, Navigation, &c ; but, that we ſhall never be able, by any means, to come at the Truth is moſt certain.

The Affinity between the Pyramid and the Priſm, and the Proportion one bears to the other, are, in the firſt place, determined ; by means of which, the Proportion between the Cone and Cylinder, is alſo determined.

Laſtly, of the Sphere ; ſhewing the Proportion it alſo has to a Cylinder and Cone ; as given by Archimedes ; for, on that Head, Euclid is totally ſilent ; the 12th Book only determines the Ratio one Sphere has to another, in the laſt Theorem. To obtain which, he has two Problems, 17th and 18th ; the latter, the moſt prolix of the whole ; containing, in four or five full Pages, the moſt tedious deſcription of the Conſtruction of a Solid, which is not of any other uſe, whatever ; and previous to it, Profeſſor Simſon alſo gives a preparatory Lemma.

The manner, in which the Ratio of one Sphere to another is here inveſtigated, and deduced from the 9th, is, at once, the moſt ſolid and convincing ; and alſo ſhews how the quantity of every Sphere is aſcertained, which the other does not ; and therefore, what it does is of little uſe.

DEFI-

DEFINITIONS.

For the Definition of a Pyramid, see Def. 7th of the 7th; which, notwithstanding it is not used there, is the first of Plane Solids (having the least number of Faces) and as a Prism (Def. 8th) includes all Parallelopipeds whatever, it could not be dispensed with, though but of little use in that Book.

It remains, only, to define such Solids as are bounded by curved Surfaces.

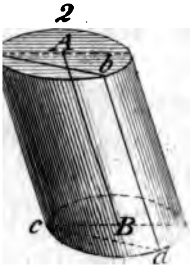
DEF. I. A CYLINDER is a SOLID, having two parallel Plane Surfaces, which are equal Circles; and an uniform convex Surface, bounded, in length, by the circular Planes, and returning into itself in width, without bounds. As AB, or *AB*.

1

DEF. II. The BASES, of a CYLINDER, are the two circular Planes; A and B.

DEF. III. The AXE, of a CYLINDER, is a Right Line passing through the Centers of its Bases. As AB, or *AB*.

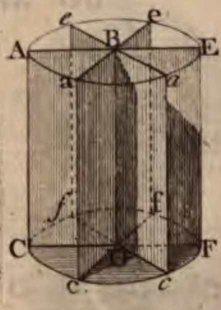
2



N. B. A Cylinder may be conceived to be generated by the Direct motion of a Circle (as A, or *A*) along its Axe, always parallel to itself.

Let the Circle A, or *A*, be supposed to be applied, at its Center, to the Right Line AB, or *AB*; and being moved, parallel to its first position, at A or *A*, to B or *B*; the Line AB or *AB*, being always in its Center; it will, in that motion, have described the Cylinder AB or *AB*.

If any Right Line as ab, or *ab*, be drawn in the Circle, cuting the Circumference, at a, or b, *a* or *b*; and, whilst the Circle described the Cylinder (the Right Line ab, or *ab*, being always parallel to its first position) the Point a or b, *a* or *b*, will describe a Right Line, ac, or bd, which is a Side of the Cylinder.



Def. IV. A Right Cylinder is one, whose Axe, AB, is perpendicular to the Planes of its Bases, A and B.

A Right Cylinder may be conceived to be generated by the revolution of a right angled Parallelogram, or Rectangle (C A B D) on one Side (BD) which remains fixed; and is therefore called its Axe, or Axis; on which the Rectangle revolves, and on which the Solid, revolving, would appear at rest.

The two Sides, AB and CD, being equal, describe the equal Bases, whilst AC describes its Surface.

Def. V. An oblique Cylinder, is when its Axe, is inclined to the Bases, (see Fig. 2)

Def. VI. A CONE is a Solid, having but one Plane Surface, which is a Circle; and a convex Surface, returning into itself; continually varying, in convexity, from the Circle which bounds it, till it terminates in a Point. As ACD, or *ACD*.

Def. VII. The Base, of a Cone, is the circular Plane CD, or *CD*.

Def. VIII. The Vertex, of a Cone, is the Point A or *A*.

Def. IX. The Axe, of a Cone, is the Right Line AB, or *AB*; from the Vertex A, or *A*, to the Center B, or *B*, of its Base.

N. B. A Cone may be conceived to be generated thus.

If CD or *CD* be a Circle, and any Point, A, or *A*, be taken out of its Plane; then, if a Right Line AB, or *AB*, be applied from the Point A, or *A*, to any Point C, or *C*; and being fixed, at A, or *A*, let it be revolved around the circumference of the Circle; in which revolution, it will describe the Cone A C D, or *A C D*.

DEF. X. A RIGHT CONE is when the Axe (AB,) is perpendicular to the Plane of its Base.

A Right Cone may be conceived to be generated by the revolution of a right angled Triangle, ABC, on either Side containing the Right Angle, as AB, which remains fixed; and is therefore called its Axe or Axis, on which it revolves.

The other Side, BC, describes the Base, in its revolution; whilst the Hypothenuse, AC, describes the Surface of the Cone, and prescribes its Bounds.

Every Section, by a Plane, through the Axe of a Right Cone, is an Isosceles Triangle, as CAD; CD is its Base; and a Right Line, AC, or AE, from the Vertex to any Point in the Circumference of its Base, is a Side of the Cone.

DEF. XI. A SCALENE or OBLIQUE CONE is when the Axe, *AB*, inclines to the Base, *CD*.

Every Section of an oblique Cone, through its Axe, is a Scalene Triangle; except in one position, when the Axe is at right angles with the Line of Section with the Base; and is, therefore, called a Scalene Cone.

When the Diameter of its Base, *CD*, is equal to the shortest Side *AD*; the Section, *CAD*, through that Side, perpendicular to the Base, is an Isosceles Triangle, and *AC*, the longest Side, is its Base; nevertheless *CAD* is called a Scalene Cone.

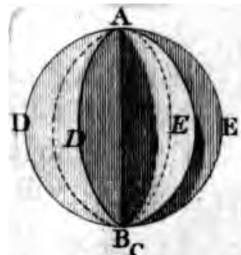
DEF. XII. A SPHERE is a SOLID, bounded by one uniform convex Surface, which has no Bounds.

As the Circle is bounded by one Line, its Circumference, so the Sphere is bounded by one Surface.

DEF. XIII. The CENTER of a SPHERE is the middle Point of the Solid; which is equally distant, every way, from its Surface. As C.

DEF. XIV.

DEF. XIV. The DIAMETER, of a SPHERE, is any Right Line paſſing through its Center, and terminated by its Surface. As A B.

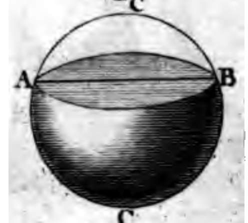


N. B. A Sphere may be conceived to be generated, by the revolution of a Semicircle; as ADB, on its Diameter AB, which remains fixed whilſt the Semicircumference revolves; and, in its revolution, deſcribes, or rather preſcribes, the Bounds of the Sphere.

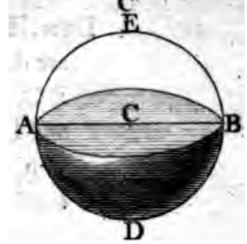
The Diameter, AB, being at reſt, on which if the Sphere revolved, it would appear at reſt, is therefore called an AXE, or Axis of the Sphere.

DEF. XV. A SEGMENT, of a SPHERE, is any portion, cut off by a Plane. As A C B.

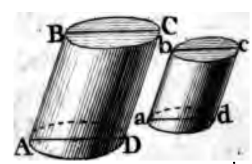
Its Baſe, AB, is a Circle, made by the Section.



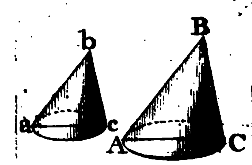
DEF. XVI. A HEMISPHERE is a Segment of a Sphere. When the Plane of the ſection paſſes through its Center, each Segment is a Hemiſphere. As ADB, and AEB, made by the Plane AB.



DEF. XVII. SIMILAR CYLINDERS, and CONES, are ſuch as have their Axes, and the Diameters of their Baſes Proportionals.



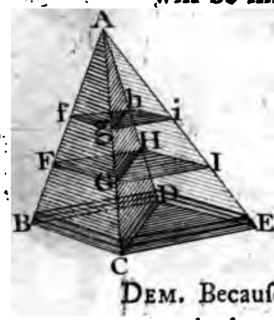
If they are ſcalene or oblique; Sections through the Axes of Cylinders, perpendicular to their Baſes, or equally inclined to their Baſes, are ſimilar Parallelograms; as ABCD, a b c d; and, in Cones, they are ſimilar Triangles; as A B C, a b c.



All Spheres are ſimilar to one another.

THEOREM I.

If a Pyramid be cut in two parts, by a Plane parallel to its Base; the lesser Pyramid, made by that Section, will be similar to the whole Pyramid.



First; let A be the Vertex, and, BCD the Base of a triangular Pyramid; ABCD.

Let FGH be a Section made by a Plane, parallel to BCD.

I say, the Pyramid, AFGH, is similar to the Pyramid ABCD.

DEM. Because the Planes BCD, FGH are parallel, and they are both cut by the Plane BAC, the Sections, FG and BC, are parallel; for the same reason, the Sections GH and CD, FH and BD are parallel. - - Th. 8.7.
Wh. AF:FB::AG:GC, and as AH:HD - - 2.6.
conf. AF:AB::AG:AC,&c.(6.5.) i.e. as FG:BC,&c-4.6.
and consequently, the Triangle AFG is similar to ABC, AGH to ACD, and AFH to ABD;
also, the Base, FGH, to the Base, BCD;
Therefore, the Pyramid AFGH is similar to ABCD.

After the same manner, the Pyramid Afgh may be proved similar to either of the other.

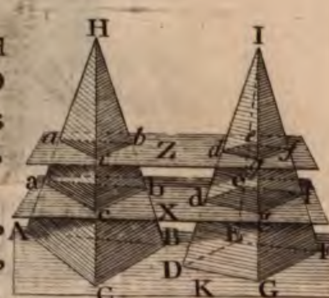
Again, Let CBDE be the Base of a quadrangular Pyramid, and, let GFHI be a Section, parallel to the Base.

It may be demonstrated, in the same manner, that each Triangle AFG, AGI, &c. is similar to ABC, ACE, &c. also, FGH is similar to BCD, and GHI to CDE; consequently, GFHI is similar to CBDE. - 13.6.

Wherefore, since a Pyramid, whose Base is a Poligon of any number of Sides, may be divided, by diagonal Sections, into triangular Pyramids, the same Demonstration will hold true in every Pyramid whatever.

THEOREM II.

Pyramids, having equal Bafes and equal Altitudes, are equal. (See the laft Figure.)



Firft; let the Pyramids A B C D and C A D E have triangular Bafes, B C D and C D E, and a common Vertex, A; confequently, they have equal Altitude,

I fay, they are equal Pyramids.

Let there be made feveral Sections, G F H I, g f h i, parallel to the Bafes, B C D, C D E, which are in one Plane.

DEM. Then, becaufe they are both cut by a Plane, parallel to their Bafes, and the Bafe BCD = CDE; FGH = GHI. For, FGH is fimilar to BCD, and GHI to CDE; - Th. 1. and each has that Ratio to the other, refpectively, which is duplicate of AC to AG; i. e. of CD to GH - 12. 6.

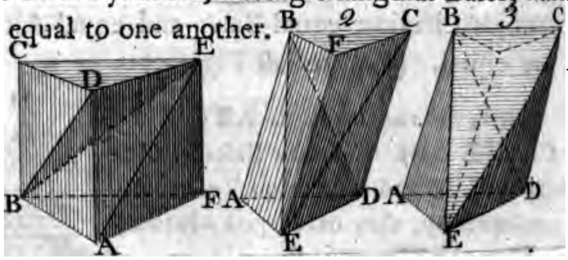
After the fame manner, the Section f g h, may be proved equal to g h i; and conf. every Section, made by Planes par. to CBDE, will be equal, each to the other, refpectively. Therefore, the Prifm B A C D is equal to C A D E.

For, the Sections being equal, every where; conf. if each Pyramid be conceived to be compofed of an infinite number of Triangular Prifms, whofe height (or thicknefs) are all equal, and the leaft that can be conceived, laid on one another, as at BCD, CDE, being each eq. to the contiguous one, the whole is eq. to the whole.

Again. Suppofe the Bafes of the Pyramids to be different in Figure, as ABC and D E F G; their Vertices, H and I, being equally diftant from the Plane of their Bafes (A K F) confequently, every Section, a b c, or *abc*, and d e f g or *defg* (made by the Planes X and Z, parallel to AKF) being in the fame Ratio, each to the other, refpectively, as the Bafes, are equal. Therefore, the Pyramid A H B C is equal to D E I F G.

THEOREM III.

Every Prism, having a triangular Base, may be divided, into three Pyramids, having triangular Bases, which are equal to one another.



First; let ACF be a Prism, whose Base ABF, and consequently CED, are right angled Isosceles Triangles.

Let the Planes, ABCD and ADEF, be Squares, and BAF, CDE, Right Angles; conf. BCEF is a Rectangle, under the Side of a Square and its Diagonal (BC and CE).

Draw, AE, BE, and BD (Diagonals in each Face) and imagine Planes to pass thro'. AE & BE, BD&BE - Ax.4.7.

The Pyramids, ABEF, AEDB, and BCED, made by the Sections of those Planes, are equal and similar.

Dem. The Triangle ADE = AEF, and ADB = BCD 15.1
But, ABCD and ADEF are Squares (by Supposition) and, they have a com. Side, AD; wh. they are equal - Ax.B.2
Also, BAF is a R. Angle (Sup.) conf. CDE is a Rt one; - 5.7.
wh. the Tri's. AFB, CED, ADB, BCD, ADE, and AEF are all equal, and similar to one another - 8. 1.
But, AEB and BED (the Planes of the Sections) are each common to two Pyramids; and they are equal and similar to each other, and also to CEB, BEF. - - 7. 1.
For, AE and BD are each equal to CE and BF; being all Diagonals; also, AB, DE, CB, and EF are Sides of equal Squares, therefore equal; and BE is common to them all.
Wherefore, each Pyramid, ABEF, AEDB, and BCED, being contained by an equal number of Planes, equal and similar to each other, respectively, are equal. - Ax. 5. 7.

For two Angles in each Pyramid, at A, C, D, and F, are contained by two R. Angles each, and half a R. Angle; as AFE, BFE, & AFB.

The remaining two, in each, meeting at B and E, are each contained by two half right Angles, (AEF, BEF, &c.) and an Angle AEB, equal DBE, (each common to two Pyramids) each equal CEB, or EBF; being contained by the Diagonal of a Square, (C E) and the Hypothenuſe (BE) of a right angled Triangle, under the Side and Diagonal, BC, and CE.

2nd. Let ABCE be a triangular Priſm; i. e. whoſe Baſes, ADE and BCF, are Triangles; ſuppoſe them Scalene, and the Sides AB, &c. of the other Faces, not perpendicular to them; the Priſm is conſ. oblique in every reſpect. The Priſm ABCE, may be divided into three eq. Pyramids.

Draw the Diagonals, BE, BD, & EC; and ſuppoſe a Plane to paſs thro'. BE & BD, and, another, thro'. BE & EC - Ax. 4. 7

The Priſm will be divided into three Pyramids; ABDE, BCEF, and BECD.

I ſay, thoſe Pyramids are equal to one another.

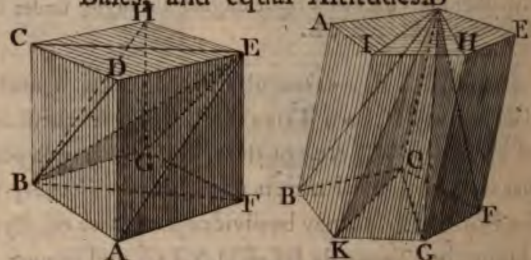
Dem. The Pyramids, ABDE and BCEF, have equal and ſimilar Baſes, ADE & BCF, and equal Altitudes; Def. 8. 7. Conſequently, every Section, made by parallel Planes, at equal diſtances from their Baſes, are equal, and alſo ſimilar; Therefore, the Pyramids are equal. - - Th. 2.

Again. Suppoſe the Pyr. BCEF (eq. ABDE) taken away; the Face BCE is common to the Pyramid BECD (ſee Fig. 3) Then, in the Pyramids ABDE, BECD; becauſe ABCD is a Parallelogram (Def. 7.) it is divided by the Diagonal, BD, into congruous Triangles, ABD and BCD. - 15. 1. Let them be conſidered as the Baſes of the two Pyramids; E being their com. Vertex, they have, conſ. equal Altitude; wh. every Section, parallel to ABCD, will be equal, and alſo ſimilar; and conſequently, the Pyramids are equal - Th. 2.

If ABDE be ſuppoſed removed (Fig. 2.) then the Pyramids BCEF, & BECD are equal; for their Baſes CEF, & CED are equal (15. 1.) and B is their common Vertex; Therefore, they are all equal to one another. Q. E. D.

THEOREM IV.

Every Prifm is triple of a Pyramid, having equal Bafes, and equal Altitudes.



First. Let the Prifm ACHF be a Cube.

Then, a Plane drawn through the oppofite Diagonals CE and BF, will cut the Cube into two equal triangular Prifms; whofe Faces FD and AC, &c. are Squares, and the Angles BAF, CDE, CHE, and BGF, are R. Angles.

Draw, AE, EB, and DB; dividing the hither Prifm, ABCEF, into three equal and fimilar Pyramids ABEF, AEDB and BCED, (as in Cafe 1ft of the 3rd.)

Alfo, draw EG. BEFG is a Pyramid, equal and fimilar to ABEF; and the Prifm ACHF is triple of the Pyramid ABEFG.

DEM. For, the Bafe BFG is equal and fimilar to ABF - 15 1.
and the Face BEG is congruous with AEB; - Ax. 7. 1.
(for EG bifects the Square FH, and AE bifects FD)
But, BEF is common; and BEG, AEB are congruous - 7. 1.
(for, BG is equal to AB, EG=AE, and BE is common)
wh. the Solid, BEFG, is contained within four triangular Planes, equal and fimilar to thofe of AEBF;
and confequently, they are equal Pyramids. - Ax. 5. 7.

But, the triangular Pyramid, AEBF, is equal one third of the Prifm, ABCEF; and BEFG is equal one third of BHF, equal CAE. - - - - Cor. 1. 3,

Therefore, the quadrangular Pyramid, ABEFG, is equal one third of the Cube ACE; or, the Cube triple of the Pyramid, on the same Base, ABGF, and having the same Altitude, EF.

nd. After the same manner, as in Case II. of the last; every triangular Prism may be proved triple of a Pyramid, having the same, or an equal Base and Altitude.

Wherefore, since every Prism, having a quadrangular Base, may be divided into two triangular Prisms; and each triangular Prism, into three equal Pyramids having triangular Bases; conf. a Pyramid on the whole quadrangular Base of the Prism, and having equal Altitude, is equal one third of the Prism, or the Prism triple of the Pyramid.

3rd. Let the Base, AIHED, of the Prism, AKFD, be a Pentagon; and let the Diagonals, CG,CK,DH,DI, be drawn.

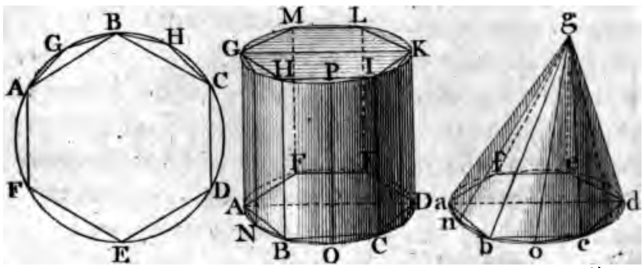
Then, because CD is parallel to AB and EF; and, GH is parallel to E F, and IK to AB - - Def. 33.
DC is parallel to I K and GH. - - Th. 4. 7.
wh. Planes may pass through DC & GH, DC & I K - Ax. 2.
and conf. the Prism, AK F D, will be divided into three triangular Prisms, by the Planes, DCKI and DCGH; viz. AKD, KDG, and DGE.

Draw the Diagonals BD, DF, DG, and DK; BDFGK is a Pyramid having the same Base, BCFGK, and Altitude as the Prism.

But, BDCK is a Pyramid, having the same Base, BCK, and Altitude, as the Prism AKD;
AKD is therefore triple of the Pyramid, BDCK.
For, the same reason, KIDHG is triple of the Pyr. KDGC; and the Prism DGE is triple of the Pyramid CDFG.
But, the Pyramid BDFGK is equal to the three Pyramids BDCK, KDGC, and CDFG; and the Prism AKFD is equal to the three Prisms AKD, KIDHG, and DGE.
Th. the Prism AKFD is triple the Pyr. BDFGK. Q.E.D.

THEOREM V.

Every Cone is equal to the third part of a Cylinder, having equal Bafes and equal Altitudes.



Let the Circle A C E be confidered as the Bafe of a Cylinder or Cone; and, let a regular Poligon, ABCDEF, of any number of Sides be infcribed; it is manifeft, that the Poligon is lefs than the Circle.

Let the Arks, AGB, &c. be bifected, as at G; and draw AG and BG, &c. This Poligon will be greater than the firft, but it is alfo lefs than the Circle.

Wherefore, it is evident, that, the more the Sides of the Poligon are multiplied, the nearer it approaches to the Circle, till the difference is lefs than any affignable Quantity whatever. The difference is the Segments AG, GB, &c.

And, by the fame reafoning, every Poligon, circumfcribed, will be greater than the Circle; confequently, they will at laft end in the Circle; i. e. the Perimeter of the Poligon, in the Circumference of the Circle, and their Areas will be equal, or the difference will be lefs than any other Quantity.

Let A G K D be a Cylinder whofe Bafe is A D; and a g d a Cone, whofe Bafe, a d, is equal to the Bafe AD, of the Cylinder; and let them alfo have equal Altitudes.

I fay, the Cylinder AGKD, is triple of the Cone, a g d.

DEM·

Dem. Let there be regular Poligons, of any number of Sides, inſcribed; as A B C D E F, and a b c d e f, whoſe Sides AB, BC, &c. a b, b c, &c. are equal;

Conſequently, they are equal Poligons.

Let a Priſm be conſtructed on the Poligon, ABCDEF, whoſe Sides AG, BH, CI, &c. are in the Surface of the Cylinder; and from every Angle of the Poligon, a b d, let there be drawn Right Lines a g, b g, &c.

Then, a b g d c is a Pyramid, whoſe Baſe is equal to the Baſe of the Priſm (byConſtruction); and, having equal Altitudes, the Priſm, AMKC, inſcribed in the Cylinder, is triple of the Pyramid, inſcribed in the Cone. - Th. 4.

Again. Let the Arks AB, BC, &c. be biſected, at N and O; and let a Priſm be inſcribed, having twice the number of Sides.

Alſo, let the Pyramid, inſcribed in the Cone, have the ſame number of Sides, as the Priſm; their Baſes are equal; and, conſequently, the Priſm is triple of the Pyramid - 4.

And ſo it muſt always be; till, by multiplying the Sides of the Priſm, its Surface will be the ſame as the Surface of the Cylinder, and the Surface of the Pyramid as the Cone; the Baſes of the Priſm and Pyramid will be the ſame as the Baſes of the Cylinder and Cone.

But, they are equal, and they have equal Altitudes.

Therefore, the Cylinder is triple of the Cone; or, the Cone equal to one third of the Cylinder. Q. E. D.

THEOREM VI.

Prisms, or Pyramids, having equal Altitudes, have that Ratio to each other, as their Bases.

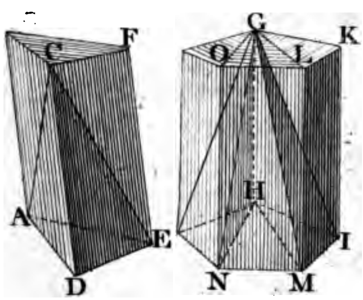
First. Let the Prisms have quadrangular Bases.

If they are Parallelopipeds, it is already proved - Cor. 1. 17. 7.

For, every Parallelopiped (is a Prism; Def. 7. 7.) may be divided, by a Plane passing through its opposite Diagonals, into two equal and similar triangular Prisms; consequently their Bases and Altitudes are equal. - - - - - Th. 16. 7.

And it is manifest, that if the Base of one be double, or triple, of the other, seeing they have equal Altitudes, they are as their Bases; i. e. the Prism, whose Base is double or triple of the others Base, is double or triple of the other Prism; and consequently, whatever ratio one Base has to the other, the Prisms have necessarily the same. (For, whether the Bases be similar Figures or not, being equal, it is the same; as it is proved, in Case the 3rd of the 17th, being Parallelopipeds.)

Also, if their Bases be equal, they are as their Altitudes-Cor. 2. 17. and, if the Prisms are equal, their Bases and Altitudes reciprocate, (Th. 19.) Consequently, every triangular Prism is to another, of equal Altitude, in the same ratio as their Bases; seeing, they are each half a Parallelopiped or Prism, whose Base is double the Base of the triangular Prism, and having equal Altitudes.



Let the Prisms AFD and EGKM have equal Altitudes; the one on a triangular Base, the other pentagonal.

Let the Diagonals GL, GO, HM, and HN be drawn, between opposite Angles in the equal Pentagons; and imagine Planes OGHN, and GLMH, drawn through those Diagonals, dividing the Prism, EGKM, into three triangular Prisms, EGN, NGM, and MGI.

DEM. Then; because the triangular Prisms, AFD and EGN, have equal Altitudes, they have that Ratio to each other, as their Bases ADE to EHN. - proved above.

i. e. the Prifm AFD:EGN: : ADE:EHN
And, the Prifm AFD:NGM: : ADE:NHM } their Bafes.
alfo, the Prifm AFD:MGI : : ADE:MHI
Wh. as the Bafe ADE:EHN + NHM + MHI
fo is the Prifm AFD:EGN + NGM + MGI } - 2. 5.
i.e. as the Bafe ADE: EHIMN : : the Prifm AFD: EGKM.

2nd. Becaufe the Prifm AFD is triple the Pyramid ACED;
and the Prifm EGKM is triple of the Pyramid NEGIM-4.
Alfo, as Bafe ADE:EHIMN,: : the Prifm AFD:EGKM
conf. as Bafe ADE:EHIMN,: : the Pyr. ACED:NEGIM

Cor. 1. Cylinders, having equal Altitudes, have the fame Ratio to each other as their Bafes.

For, fimilar Poligons, infcribed in Circles, are, to each other, as the Squares of their Diameters. - - - Th. 14. 6.

And it has been proved, that all Prifms, having equal Altitudes, are in the Ratio of their Bafes, to each other; confequently, Cylinders are to each other in the fame Ratio.

Becaufe, every Cylinder, whether it be right or oblique, is equal to a Prifm, whofe Bafe is the fame, or equal to the Bafe of the Cylinder; i. e. having its Sides multiplied continually till it ends in the Surface of the Cylinder; and its Bafe, in the Bafe of the Cylinder.

Cor. 2. Cones, having equal Altitudes, are to each other, in the Ratio of their Bafes.

Becaufe, a Cone is equal to one third of a Cylinder, having equal Bafes and Altitudes; and Cylinders are in that Ratio to each other; therefore Cones have the fame Ratio as Cylinders.

Cor. 3. Prifms and Pyramids, having equal Bafes, are to each other as their Altitudes.

Becaufe Parallelopipeds are Prifms; and it has been proved, that Parallelopipeds are to each other in that Ratio; and confequently, all Prifms, having equal Bafes, are as their Altitudes.

Cor. 4. Cylinders and Cones, having equal Bafes, are to each other as their Altitudes.

Becaufe, there is the fame Ratio between Cylinders and Cones, as between Prifms and Pyramids, having equal Bafes. Therefore, &c.

2

THEOREM VII.

Similar Prifms and Pyramids are, to one another, in th triplicate Ratio of their correfponding Sides. *

In refpect of Prifms (being alfo Parallelopipeds) this property is already proved (in Theo. 18th of the 7th) and confequently, in Prifms having triangular Bafes; feeing, every fuch Prifm is half a Parallelopiped, whofe Bafe is double of the Prifm and having equal Altitudes.

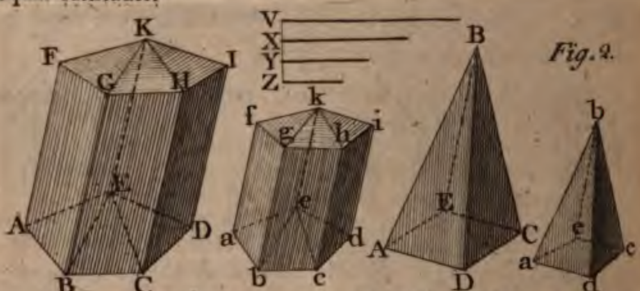


Fig. 2.

Let the Pentagonal Prifms A K D, a k d be fimilar.

Take any Right Line, V; and as a b is to A B, &c. make X to V; and as X is to V, fo make Y to X; alfo, as Y is to X, or X to V, make Z to Y.

I fay, that, as V is to Z, fo is the Prifm AKD to akd.

Let them be divided by diagonal Planes, BK, KC, bk, kc, paffing through the oppofite Diagonals; dividing the Prifms into three triangular Prifms, each.

* I cannot conceive the reafon, why Euclid confines this proportion, and the following, to Pyramids only; and, to fuch, only, as have triangular Bafes; feeing that, it holds equally true in Prifms; and alfo in Pyramids, having poligonal Bafes, of any kind.

DEM.

DEM. The Poligons ABCDE, abcde, are ſimilar - Hyp.
wh. the Triangles ABE, a b e, &c. are ſimilar. - 13. 6.
and conſ. the Priſms AKB, akb, &c. are ſimilar- Ax. 11.7
(For, as AB:ab, or as BC:bc, : : BG:bg, and GK : gk ;
conſequently, the diagonal Planes, BK and bk, KC
and kc, are ſimilar).

Wherefore, ſince each triangular Priſm is equal to half a Parallelopiped, whoſe Baſe is double of the triangular Priſm, and Altitudes equal (above) they have that ratio to each other, which is triplicate of their correſponding Sides, i. e. as V to Z. (Ax. 8. 5.) - - - Th. 18. 7.
But the Priſm AKB:akb: : BKC:bkc, and as CKD:ckd.
wh. as AKB:akb: : AKB+BKC+CKD:akb+bkc+ckd.*
i. e. as the Priſm AKB : akb : : the Priſm AKD : akd.
But, AKB:akb: : V:Z, Th. the Priſm AKD:akd: : V:Z.†

2nd. Let the Pyramids ABCD, abcd be ſimilar. (Fig. 2.)
The Pyramids ABCD, abcd, being ſimilar, are each equal to the third part of a Priſm on the ſame Baſe and Altitude.

But, Quantities are in the ſame Ratio, to each other, as their Equimultiples or equal Parts. - - Ax. 8. 5.
and the triples of the Pyramids ABCD, abcd, are as V to Z.
therefore, the Pyramid ABCD : abcd : : V:Z. Q. E. D.

COR. Similar Cylinders are, to one another, in the triplicate Ratio of their Diameters or Sides.

For, they are in the ſame Ratio, to each other, as ſimilar Priſms inſcribed ; or, as Priſms, whoſe Baſes are equal to the Baſes of the Cylinders, and having equal Altitudes.

COR. 2. Similar Cones are, to each other, in the triplicate Ratio of the Diameters of their Baſes, or of their Axes.

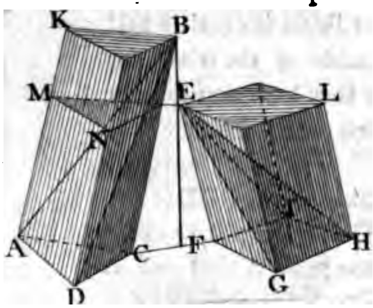
Becauſe a Cone is equal to the third part of a Cylinder, on equal Baſes and Altitudes.

* By Theo. 2nd of the 5th. † By Axiom 13 of the 15th.

THEO-

THEOREM VIII.

Equal Prisms, or Pyramids, have their Bases and Altitudes reciprocally proportional.



If the Prisms are Parallelopipeds it is already proved. - - 19.7.

Let the triangular Prism AKC be equal to the quadrangular Prism FLG; whose Bases, ACD, FGHI, are in the same Plane; to which, Draw the Perpendicular BF. BF and EF are their Altitudes.

I say, the Base, ADC, is to the Base, FGHI, as EF is to BF.

Let the Plane of the Top, LE, be produced, cuting the triangular Prism, at EMN.

Dem. The Prism AMC:AKC::EF:BF, their Altitudes. And - - - AMC:FLG::ADC:FGHI, their Bases. But, the Prism FLG is equal to AKC; by Hypothesis, consequently, AMC:FLG::EF:BF, i.e. as ADC:FGHI Therefore, as ADC:FGHI :: EF : BF. Q. E. D.

2nd. Draw the Diagonals AB, DB; EG, EH, and EI.

The Pyramid ABCD is equal to one third of AKC -4. And, the Pyramid FGEHI= one third part of FLG. But, Quantities are in the same Ratio as their Equimultiples; wherefore, the the Pyramid ABCD is equal to FGEHI.

But, they have the same Bases and Altitudes, as the Prisms, AKC and FLG.

Th. their Bases and Altitudes are reciprocally proportional.

Cor.

COR. Equal Cylinders, or Cones, have their Bafes and Altitudes reciprocally Proportional.

Becaufe, Cylinders are equal to Prifms whofe Bafes and Altitudes are equal; and Cones to Pyramids.

In thefe eight Theorems, and their Corollaries, with the Corollary to the next, is contained the whole of Euclid's 12th Book; thofe which follow, are felect Theorems from Archimedes; which fhews how the furface of the Sphere may be afcertained: alfo, of any Segment, or Portion of its Surface, intercepted between parallel Circles, with great accuracy.

The next Theorem, which is not from Archimedes, fhews, in a brief and elegant manner, the Ratio between the Sphere and a circumfcribing Cylinder; and confequently, if the quantity of the Cylinder be known, or attainable, the quantity of the Sphere may alfo be obtained. According to Archimedes, it is determined by the Cone, in the 17th of this Book; which, though perfectly true, is not fo brief; and though perhaps more geometrical, yet not more convictive and fatisfactory.

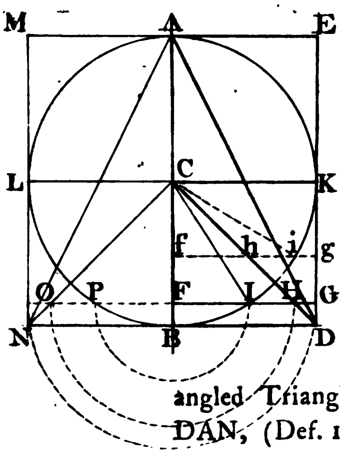
In the following is contained all that is really ufeful or valuable, in refpect of the Cylinder, Cone, and Sphere, nor is it poffible, I think, to obtain the Area of the Surface of a Sphere in a more concife manner.

In fhort, without the following Theory, I cannot but think the Doctrine of Solids imperfect; feeing that, the Quantity of a Sphere, the moft perfect Solid, cannot, by Euclid, be obtained, in refpect either of its Superficies or folid Contents; for want of which, Artificers are frequently at a lofs, in meafuring a Dome or other fpherical Object.

THEOREM IX.

Every Sphere is equal to two thirds of a circum-
ſcribing Cylinder; i. e. of a Cylinder whoſe Dia-
meter and Altitude are equal, each to the Diameter
of the Sphere.

Let AB be the Diameter of a Sphere; biſect AB, at C.



On C, deſcribe the Semicircle AKB; and, on A B, conſtruct the Rectangle AEDB, circumſcribing it, and draw AD.

Then (AB remaining fixed) imagine them all revolved about AB, as an Axis; the Semicircle, A K B will deſcribe a Sphere, A K B L, (Def. 14.)

The Rectangle, A E D B, will deſcribe a Right Cylinder, D E M N, containing the Sphere (Def. 4.) and, the right-angled Triangle, ADB, will generate a Right Cone, DAN, (Def. 10) whoſe Baſe, D N, is the ſame as the Cylinder, and equal to the Diameter of the Sphere.

The Cone, D AN, is equal to one third part of the Cylinder. - - - - - - Th. 5.

I ſay, that the Sphere, A K B L, is equal to the remaining two thirds.

Draw C D; and F G (at pleaſure) parallel to C K; cuting C D at H, and the Circumference of the Sphere at I; and draw C I.

DEM. In the Rt. angled Tri. CFI, $CI \square = CF \square + FI \square$ - 20.1
and, the Areas of Circles are as the Squares of their Diameters. - - - - - - C. 1. 14. 6.
wherefore, a Circle, deſcribed on C I, is equal to two Circles, on CF and F I.

Now, in the revolution of the Square CKDB, there will be defcribed the Cylinder KDNL; the Ark BIK will defcribe a Hemifphere; and, CD will defcribe the Surface of a Right Cone, DCN; alfo, the Points G, H, and I, will defcribe Circles, with the feveral Radii FG, FH, and FI; FG is equal to CI (equal CK) and FH is equal to CF. - - - - 9.1. (for, the Angle CFH is Right, and the Angles FCH, FHC, are half Right; feeing that, the Diagonal, CD, bifects the Angles BCK, KDB)

Wherefore, the Circle defcribed by FH (equal CF) is equal to the difference between the Circles defcribed by FG (equal CI) and FI.

Confequently, the Circle, defcribed by FH, is equal to the Annulus or Ring, defcribed by GI;

and, confequently, every Section (FG or fg) of the Cone DCN, by a Plane parallel to its Bafe, is equal to a Section of the concave Solid, remaining when the Sphere is fuppofed to be taken out of the Cylinder.

Wherefore, the Cone, DCN, is equal to the Solid defcribed by the mixed Triangle KiIBD.

But, the Cone, DCN = one third of the Cylinder KDNL conf. the concave Solid (taking away the Hemifphere, KBL, out of the Cylinder, KDNL) is equal to one third of that Cylinder; and confequently, the Hemifphere, KBL, is equal to two thirds of the Cylinder.

But, the Cone, DCN, is half the Cone DAN; and, the Cylinder, DKLN, is half the Cylinder DEMN; C. 4.6. alfo, the Sphere, AKBL, is double the Hemifphere, KBL. Therefore, the Sphere AKBL, whofe Diameter, AB or KL, is equal to the Diameter and Altitude of the Cylinder, DEMN, is equal to two thirds of the Cylinder. Q.E.D.

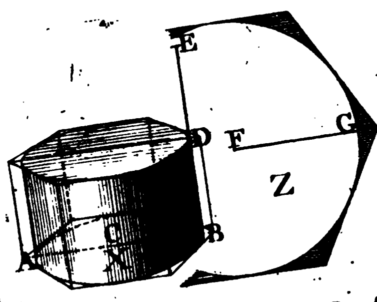
COROLLARY. The 18th Propofition, of Euclid.

Spheres have that Proportion to one another, which is triplicate of their Diameters;

For, all Cylinders, being similar, are in that Ratio to one an[illegible] o-ther; and consequently, Spheres, inscribed in Right Cylinder[illegible]rs whose Diameters and Altitudes are equal, respectively, to the Di[illegible]a-meter of the Spheres, being two thirds of such Cylinders, are i[illegible] in the same Ratio to each other as the Cylinders; i.e. in the tripl[illegible]li-cate Ratio of their Diameters.

THEOREM X.

A Circle, whose Radius is a mean Proportional be-tween the Diameter of the Base, and the Side of a Right Cylinder, is equal to the cylindrical Surface.



Let AB be the Diameter of the Base of the Cylinder AD, and let BD be its Side.

Take FG a mean Proportional, between AB and BD, (Prob. 30) and on FG, de-scribe a Circle; let C be the Center of the Base of the Cylinder; produce BD, and make DE equal to BD.

Then, CB, FG, & BE are Proportionals; - - - Ax. 8. 5.

for, CB : AB :: BD : BE, - - 9. 6.

wherefore, CB $\times$ BE = AB $\times$ BD - - Cor. to 9. 6.

i. e. to FG □; for AB $\times$ BD = FG □.

Let similar Poligons (X and Z) be circumscribed abou the Base of the Cylinder, and the Circle whose Radius is FG and let a Prism, be supposed to circumscribe the Cylinde

DEM. A Triangle, whose Base is equal to the Perimeter the Poligon, and its Altitude equal CB, is equal to t Poligon;* and a Rectangle under the Perimeter and BD Side of the Prism, is equal to the Superfices of the Prism-Also, a Triangle under the Perimeter and BE is equa a Rectangle under the Perimeter and BD, half BE.

* This follows from the 18th of the first. For, all the Si the Poligon are equal; and all Triangles having equal Bas Altitudes are equal. (See Art. 7. P. 7. in the Appendix).

But, a Triangle, under the Perimeter and C B, is to a Triangle under the fame Perimeter &BE, as CB is to BE-1.6 conf. the Poligon is to the Surface of the Prifm, as CB:BE. But, CB to BE is duplicate of CB to FG - Def.12.5. wherefore, the Poligon X is to the Superficies of the Prifm, in the Duplicate ratio of CB to FG.

But, the Poligon X is to Z in the Duplicate ratio of their Sides, or Perpendiculars, CB to FG, i. e. as CB to BE. wh. the Poligon X has the fame Ratio to Z, as X has to the Superficies of the Prifm, AD; conf. they are equal-Ax.5.5

After the fame manner, it may be proved that the Superficies of any other Prifm, whofe Faces being multiplied infinitely, till they end in the Surface of the Cylinder, AD, is equal to the Poligon circumfcribing the Circle, Z, whofe Sides being multiplied, end at laft in the Circle.

Therefore, the Circle, Z, whofe Radius is a Mean, between AB &BD, is equal to the Surface of the Cylinder, AD.

COR. 1. The Superficies of a Right Cylinder is to its Bafe, as the Side of the Cylinder to the fourth part of the Diameter of the Bafe.

For the Superficies of the Cylinder is equal to the Circle, whofe Radius is FG; and that Circle, is to the Bafe of the Cylinder in the duplicate ratio of F G to C B; i. e. of B E to C B; i. e. of B D to half C B.

COR. 2. The Superficies of a Cylinder whofe Side is equal to the Diameter of its Bafe, is equal to four times the Area of the Bafe; conf. if the Side be a fourth part of the Diameter the cylindrical Superficies, and the Bafe are equal.

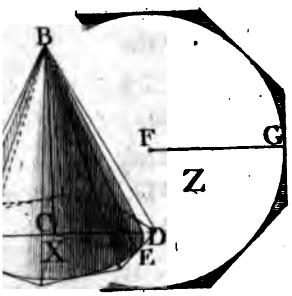
COR. 3. The Superficies of a Right Cylinder is equal to a Rectangle, under its Circumference and Side.

COR. 4. Cylindrical Surfaces, of equal height, are to each other as the Diameters of their Bafes; and, having equal Bafes, they are as their Altitudes.

COR. 5. The Surfaces of fimilar Cylinders have that Ratio between them; which is duplicate of their Diameters.

THEOREM XI.

A Circle, whose Radius is a mean Proportional between the Side of a Right Cone and the Radius of the Base, is equal to the Superficies of the Cone.



Let ABD be a Right Cone, the Diameter of whose Base is AD, and C its Center.

Take FG, a mean Proportional between the Side, BD, and the Radius CD (equal BE and CE) and on FG describe a Circle.

Let regular Poligons (X and Z) be circumscribed about the Base of the Cone and the Circle, whose Radius is FG, of the same kind; and suppose a Pyramid, on the Poligon X circumscribing the Circle.

DEM. Because CE : FG :: FG : BE; by Hypothesis.

CE to BE is duplicate of CE to FG. - Def. 12. 5.

But, as CE is to BE, so is the Triangle, whose Altitude is CE, and its Base, the Perimeter of the Poligon X, to the Triangle under BE and the same Perimeter. -Th. 1. 6.

Also, as CE is to BE so is the Triangle under CE and the Perimeter of the Poligon X, to a Triangle, under FG and the Perimeter of Z. - - Th. 6. and 12. 6.

But, a Triangle, under CE and the Perimeter of X, is equal to the Poligon X; and, the Triangle under FG and the Perimeter of Z is equal to that Poligon; also, the Triangle under BE and the Perimeter of X, is equal to the Superficies of the Pyramid.*

* This also follows from the 18th of the first Book of Elements, and Ax. 1. 1. (See the Reference to the last).

Wherefore,

Wherefore, the Poligon X has the ſame Ratio to the Poligon Z, and to the Superficies of the Pyramid; conſequently, the Poligon Z is equal to the Superficies of the Pyramid. - - - - - - - Ax. 5. 5.

After the ſame manner it may be proved, that the Superficies of any other Pyramid (whoſe Baſe is a regular Poligon, ſimilar, to one circumſcribing the Circle, whoſe Radius is F G, till by multiplying the Sides, equally, the Perimeters, of both, fall into the Circumferences of the Circles, and the Superficies of the Pyramid into the Cone) is equal to the Poligon Z; conſequently, the Superficies of the Cone is equal to the Circle whoſe Radius is FG, a mean Proportional between the Side BD & the Radius CD.

COR. 1. The Superficies of a Right Cone is to its Baſe, as the Side of the Cone to the Radius of the Baſe.

For, they are, reſpectively, equal to Triangles, whoſe Baſes are equal to the Circumference of the Baſe, and Altitudes equal to BD and CD reſpectively; conſequently, they are to each other, as BE to CE. - - - - - - - 1.6.

COR. 2. The Superficies of a Right Cone is equal to a Triangle, whoſe Baſe is equal to the Circumference of the Baſe of the Cone, and its Altitude equal to a Side of the Cone.

Hence it is manifeſt, that the Surfaces of Right Cones have all the ſame Properties as Triangles.

If their Sides be equal, they are, to each other, as the Diameters of their Baſes; and having equal Baſes, they are as their Sides; or as their Axes.

If they are ſimilar, the Ratio between their Surfaces is duplicate of their Sides, or Diameters of their Baſes.

If they are equal, their Sides and Diameters are reciprocally Proportional.

COR. 3.

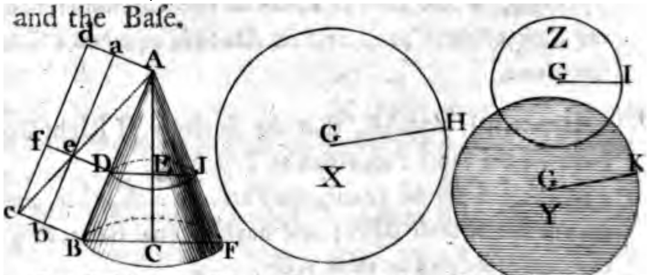
COR. 3. The Superficies of a Right Cylinder, is to a Right Cone, having equal Bases and Altitudes, as the Side of the Cylinder, to half the Side of the Cone.

For, the Superficies of the Cone is to its Base, as the Side to the Radius; or, as half the Side to half the Radius. And the Superficies of the Cylinder, to its Base, is as the Side to half the Radius; therefore, the Surface of the Cone is to the Surface of the Cylinder, as the Side of the Cylinder to half the Side of the Cone.

COR. 4. An Equilateral Cone, i. e. whose Side is equal to the Diameter of its Base, has it Superficies double of its Base; and if the Vertex be right angled, the Ratio is as the Diagonal of a Square to its Side.

THEOREM XII.

If a Right Cone be cut by a Plane, parallel to its Base; a Circle whose Radius is a mean Proportional, between part of the Side (betwixt the Base and the Section) and the Radius of that Section added to the Radius of the Base, is equal to the conical Superficies, which is between the Section and the Base.



Let BAF be a Right Cone; and let DJ be a Section, parallel to the Base, BF.

Take GH a mean Proportional, between AB, and BC; and GI a Mean between AD and DE; on which, describe Circles, X and Z.

The Circle X, on GH, is equal to the whole Superficies; and, that on GI = to the Surface of the Cone, D A J - 11. and, AB × BC = AD × DE, + DB × BC, + DB × DE.*

Take GK a Mean, between DB and DE + BC; on which, describe a Circle; let it be called Y.

I say, that Circle (Y) is equal to the conical Superfices B D J F; between the parallel Circles.

Dem. Because GH is a Mean, between AB, & BC - by Con. the Square of GH = AB × BC; i.e. to the Rect. AB cd; } 9.6
For the same reason, GI □ = AD × DE; i.e. to AD ea }
And, because GK is a Mean, betwixt BD and DE + BC; GK □ = BD × BC, + BD × DE; (equal Bf + Be) i. e. to a Rectangle under BD and DE + BC.
Consequently, GH □ = GI □ + GK □.
(Because, the Rectangle equal to GH Square, is equal to two Rectangles, which are respectively equal to the two Squares, of GI and GK)

But, Circles are to each other, as the Squares of their Diameters, and consequently as their Radii - C. 1. 14. 6. And the Circle X is equal to the whole conical Surface; also, the Circle Z is equal to the Surface D A J. Therefore, the Circle Y (being equal to the Difference between the two, X and Z) is equal to the Superficies B D F J; between the Section, DJ, and the Base.

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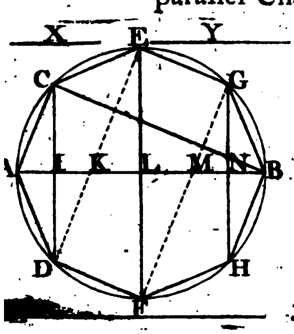
* This is deducible from the second Book of Elements; having formed a Construction.

Let ABcd be a Rectangle, under AB and BC; Ac being drawn, and Df parallel to Bc, cuting it at e; and ab (through e) parallel to AB.

Then, the whole, Bd (equal AB × BC) = Da (equal AD × DE) + BF (equal BD × BC) + cd (equal BD × DE, i. e. Be, 19. 1.)

THEOREM XIII.

If a regular Poligon, having an equal number of Sides, be inscribed in a Circle, and if a Diameter (which is also a Diagonal of the Poligon) be drawn, and another Diagonal, from either extreme of the Diameter to the adjacent Angle; the Rectangle, under the Diameter and that Chord, is equal to a Rectangle under a Side of the Poligon and all the parallel Chords, joining two and two contiguous Angles.



In the Circle AGF, describe a regular Octagon, ACEGBHFD; draw the Diameter AB, and the Diagonal BC; also, draw CD, EF, and GH.

I say, the Rectangle, under AB and BC, is equal to a Rectangle under AC (or AD) and CD, added to EF, added to GH.

Join DE and FG.

DEM. Because CE $=$ DF, and EG to FH, the Chords CD, EF, and GH, are all parallel - Cor. to 10. 3. And, for the same reason, AC, DE, FG, & HB are parallel; conf. the Triangles ACI, IDK, KEL, &c. are similar. Wh. AI:IC::IK:ID,::KL:LE,::LM:LF, &c. - Th. 4. 6. i. e. as AI : IC :: AB : CD $+$ EF $+$ GH, - 2. 5. for, the Diameter, AB, $=$ AI $+$ IK $+$ KL, &c. and CD $+$ EF $+$ GH $=$ CI $+$ ID $+$ EL, &c. - Ax. 1. 1. But, as AI:IC::AC:CB (7. 6) for ACB is a R. Angle - 12. 3. Consequently, AC:CB::AB:CD$+$EF$+$GH - Ax. 13. 5. Therefore, AC$\times$CD$+$EF$+$GH$=$CB$\times$AB - Th. 9. 6.

COR. Hence it is manifest, that if in any Segment of a Circle, as GAH, a Poligon be inscribed, on GH, whose Sides are equal, and equal in Number; a Rectangle, under the Chord CB and AN (part of a Diameter, perpendicular to GH) is equal to a Rectangle under a Side AC, and the Chords, CD $+$ EF $+$ GN, half the Base.

THEOREM XIV.

If a regular Poligon be inſcribed in a Circle, (as in the foregoing Theorem) and Chord Lines be drawn, (as there deſcribed); a Circle, whoſe Radius is a mean Proportional between the Diameter (AB) and the Chord CB, is equal to all the conical Superficies deſcribed by the revolution of the Poligon, on the Diameter or Axis, AB.

The ſame Conſtruction remaining, as in the former Figure.

Find X a mean Proportional between AC and CI; alſo Y, a Mean, between CE and CI added to EL.

DEM. Now, AC, CE, EG, & GB are equal, by Conſtruction. And, $AB \times BC = AC \times CD + EF + GH$ (Th. 13.) i. e. to $AC \times CI, + CE \times CI + EL, + EG \times EL + GN, + GB \times GN$

Find Z, a mean Proportional between AB and BC; the Square of Z is equal to the Squares of X and Y, taken twice; i. e. to $AC \times CD, + CE \times EF, + EG \times GH$ - 13. On X and Y deſcribe Circles; the Circle X is equal to the Surface of the Cone DAC, - - - Th. 11. and, the Circle Y is equal to the conical Surface ECDF - 12.

But, Circles are in the ſame Ratio as the Squares of their Diameters, (C. 1. 14. 6.) wherefore, a Circle, deſcribed on Z, is equal to X and Y, taken twice.

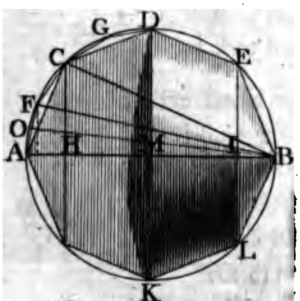
But, the Cone $DAC = GBH$; and $ECDF = EGHF$; for, CD is equal to GH, and CE to EG, &c. Therefore, a Circle (Z) whoſe Radius is a mean Proportional between AB and BC, is equal to all the conical Superficies, deſcribed by the revolution of ACEGB, on AB.

COR. After the ſame manner it may be proved, that a Circle, whoſe Radius is a mean Proportional between CB & AN, is equal to all the conical Superficies deſcribed by ACEG on AN, in the Segment GAH.

THEOREM XV.

The Surface of a Sphere is quadruple of the greateſt Circle, made by a Section through its Center; i. e. to a Circle whoſe Radius is equal to the Diameter of the Sphere.

Let ADBK be a Circle, the Section of a Sphere through its Center.



Let there be inſcribed a regular Poligon, having an even number of Sides; draw the Diameter, AB, and the Diagonal BC, to the adjacent Angle.

Imagine the Circle ADBK, with the Poligon, revolved on AB; the Circle will generate a Sphere (NB. Def. 14.) and the Sides of the Poligon AC, CD, &c. will generate Right conical Surfaces, inſcribed in the Sphere.

AC and EB will deſcribe equal Cones (for they are equal, and equally inclined to the Axe AB); CD and DE will alſo deſcribe equal Surfaces; for they are equal, and CH is equal to EI.

The whole conical Superficies, is leſs than the Superficies of the Sphere; for the Arks, AFC, CGD, &c. are greater than the Chords AC, CD, &c; alſo, a mean Proportional, between AB and BC, is leſs than the Diameter, AB.

Let the Arks, AC, CD, &c. be biſected, at F and G; and draw AF, FC, &c.

It is manifeſt, that the conical Surfaces, deſcribed by the revolution of AFCGD, will be greater than thoſe deſcribed by AC, CD; but they are leſs than the Surface of a Hemiſphere, deſcribed by the Ark ACD; and, a mean Proportional between AB and BF, is greater than a Mean, between AB and BC, but it is alſo leſs than the Diameter AB.

By the ſame reaſoning, i. e. by multiplying the Sides of the inſcribed Poligon, the Superficies of Cones, inſcribed in the Sphere, will, at laſt, end in the Surface of the Sphere; when the Sides, of the Poligon end in the Circumference of the Circle. For, if AF be biſected, at O, the Chords AO, OF diviate but very little from the Arks they ſubtend; and a mean Proportional between AB and BO, will differ very little from the Diameter AB; and conſequently, it will at laſt be the ſame as the Diameter.

DEM. But, a Circle, whose Radius is equal to a Mean, between AB and BF, or between AB and BO, is equal to all the conical Superficies, described by the revolution of the Poligon, whose Sides are equal to AF or AO-Th. 14.

Consequently, a Circle whose Radius is equal to the Diameter, AB, is equal to the Surface of a Sphere, described by the revolution of the Semicircle ADB.

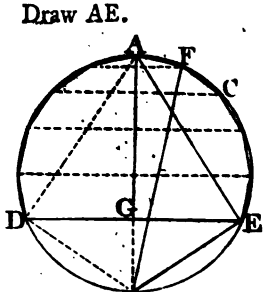
But, Circles are to each other, or amongst themselves, as Squares of their Diameters, - - - Cor. 1. 14. 6. and the Square of any Right Line is equal to four times the Square of half that Line, - - Cor. to 4. 2. Conf. a Circle whose Radius is equal to the Diameter of another, is quadruple, i. e. equal to four times that Circle.

Th. the Superficies of a Sphere is equal to four great Circles, equal, in Diameter, to the Diameter of the Sphere.

COR. The Superficies of the Segment of a Sphere is equal to four Circles, whose Diameter is a Right Line drawn from the middle Point in its Surface (i. e. the Point where a Perpendicular, from the Center of its Base, cuts it) to the Circumference of its Base.

For, by the preceding, it is proved, that the conical Superficies, in every Segment, is equal to a Circle, whose Radius is equal to a mean Proportional, between the Chords CB, FB, or OB and AI, the Altitude of the Segment EAL.

Let DAE be a Segment, and AG its Axis. Draw AE.



Then, because all the conical Surfaces, described by AF, FC, &c. is equal to a Circle, whose Radius is a mean Proportional between AG and BF; conf. by multiplying the Sides of the Poligon, BF will coincide with AB.

But, AE is a Mean Proportional between AB and AG. - - - Cor. 2. 7. 6. Consequently, a Circle, whose Radius is AE, is equal to the sphericalSurface, described by the revolution of DAE.

COR. 2. The Superficies of a Right Cylinder, whose Diameter and Altitude are equal, each to the Diameter of a Sphere, is equal to the Superficies of the Sphere.

For, the cylindrical Surface, being equal to a Circle whose Radius is a mean Proportional between its Side and Diameter, which are in this Case equal (Theo. 10.) is equal to four times its Base, i. e. to a great Circle of the Sphere.

But, the Surface of the Sphere is also equal to four such Circles. Therefore, the cylindrical Surface is equal to the spherical-Ax. 3. 1

THEOREM XVI.

If a Sphere be inſcribed in a Cylinder, and being both cut by Planes, parallel to the Baſes of the Cylinder; each Segment of the ſpherical Surface will be equal to the portion of the cylindrical, intercepted between the ſame parallel Planes.

Let the Sphere, AEBF, be circumſcribed by the Cylinder GIKH, and let them be both cut by the Planes CD, EF, parallel to the Baſes, GH, IK, of the Cylinder.

I ſay, the ſpherical Surface NAO, or ENAOF, is equal to the cylindrical Surface ICDK, or IEFK; alſo, the ſpherical Surface NEFO, is equal to the cyl. Surface, CEFD.

Let AB, be the Axe of the Cylinder, and Sphere; and let the Diameters CD, EF, of the Cylinder, be drawn, cuting the Axe perpendicularly, at L and M, and the Surface of the Sphere at N and O, E and F. Draw AN, NB, and AE, EB.

DEM. Then becauſe ANB is a Right Angle (12. 3.) AN is a mean Proportional between AL and AB; i. e. between CI and CD. - - - - - C. 2.7.6.
Alſo, becauſe AEB is a Right Angle, AE is a Mean, between AM and AB, i. e. between EI and EF. - ſame.

But, the Circle, whoſe Radius (AN) is a mean Proportional, between CI and CD, is equal to the cylindrical Superficies, ICDK. (10.) And, it is equal to the Superficies of the ſpherical Segment NAO. Therefore, the Superficies of the two Segments, of the Cylinder and Sphere, are equal.

After the ſame manner it may be proved, that the Superficies IEFK, of the Cylinder, is equal to the Surface of the Segment, ENAOF.

Conſ. the Surface of the Cylinder, CEFD, between the parallel Circles, CD and EF, is equal to the Segment of the ſpherical Surface, NEFO, between the ſame parallel Planes.

And, it is alſo manifeſt, that the remaining Superficies, EGQHF, of the Cylinder, is equal to EBF of the Sphere.

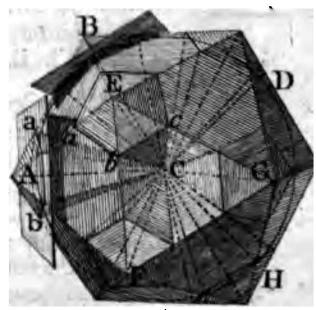
From this and the foregoing moſt excellent Theorems, is deduced a certain Rule for aſcertaining the true Area of the Surface of any Segment of a Sphere or other Portion, contained between parallel Circles.

COR. The Segments of a ſpherical Surface, intercepted between parallel Circles, have the ſame Ratio, between themſelves, as the Segments of the Axe, or Diameter (AB) cut by theſe Planes.

For, each Segment, or portion of the ſpherical Surface, NAO, NEFO, and EBF, is reſpectively equal to the correſponding portion of the Cylinder, ID, CF, and EH; and thoſe Portions are, to each other, as IC to CE, to EG, i. e. as AL:LM:MB; conſequently, the Surfaces, NAO, NEFO, and EBF, are to each other, as AL, LM, and MB.

THEOREM XVII.

A Sphere is equal, in its ſolid Contents, to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius.

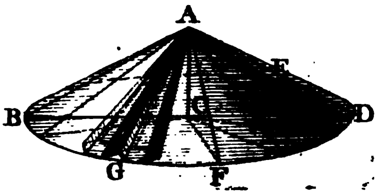


Imagine the Sphere ADF circumſcribed by any regular Body whatever; and, from every Angle of the circumſcribing ſolid ſuppoſe Right Lines, AC, EC, &c. drawn to the Center, C; there will be formed as many Pyramids, as the Solid has Faces, whoſe common Vertex is at the Center of the Sphere; which, together, is greater than the Sphere.

Imagine further, that every Angle, A, B, E, &c. of this Solid, be cut off by a Plane, touching the Sphere, and perpendicular to the Right Line from the Angle cut off. It is evident, that there will be formed another regular ſolid, containing the Sphere, touching every Face; which Solid is leſs than the firſt, but it is greater than the Sphere; and, by drawing Lines from every Angle, as before, there will be formed as many Pyramids as this Solid has Faces.

By proceeding thus, *ad infinitum*, it is manifeſt that the exterior Surfaces of this Solid, which conſtitute all the Baſes of the innumerable Pyramids, muſt, at laſt, end in the Surface of the Sphere; and conſequently, the Pyramids, or whole Solid, is the ſame as the Sphere; or, the difference leſs than any given Quantity.

Then, let BEF be a Circle, and C its Center. If the Radius BC be equal to the Diameter of the Sphere, the Circle BEF is equal to the Surface of the Sphere - Th. 15.



Make AC (perpendicular) equa[l to] its Radius, and imagine the R[i]g[ht] angled TriangleABC revolved aro[u]nd, the Hypothenuse, AB, will describe the Surface of a Cone, BADF.

I say, the Cone is equal to the Sphere.

DEM. For, because the Base of the Cone is equal to the Surface of the Sphere, it may be divided into as many plane Figures, as the circumscribing Solid has Faces; each equal, respectively, to the other.

Consequently, if Right Lines be drawn from every Angle of those Figures, to the Vertex (as at G) there will be formed as many Pyramids, as the Solid has Faces.

But, they are, respectively, equal to them, by Hypothesis; and they have, also, equal Altitudes; i. e. the Radius of the Sphere. Conf. each Pyramid is equal to its corresponding one in the Solid; and, conf. all the Pyramids contained in the one are equal to the other. But, Cones and Pyramids, whose Bases are equal to one another, are equal. Th. the Cone, BADF, is equal to the Sphere ADF.

COR. 1. Sectors of Spheres, * are equal to Cones, whose Bases are equal to the spherical Superficies of the Sectors, and the Altitude of the Cone to the Radius of the Sphere.

For, since the whole Sphere is equal to a Cone, whose Base is equal to the whole Surface, and Altitude to the Radius; conf. a Cone having the same Altitude will be to the other, as the Base of one to the Base of the other; i. e. as the portion of the spherical Surface to the whole; and, conf. the Sector is equal to such a Cone.

COR. 2. A Hemisphere is double of a Cone, having the same Base and Altitude.

For, because the Surface of a Sphere is equal to four large Circles; the Surface of the Hemisphere is double its Base.

But, a Cone whose Base is equal to the spherical Superficies of the Hemisphere, and equal Altitude, is equal to the Hemisphere; consequently, if the Base of the Cone be equal to the Base of the Hemisphere, it will be half the Hemisphere.

COR. 3. A Segment of a Sphere is equal to the Difference between a Sector, whose spherical Superficies is the Surface of the Segment, and a Right Cone, whose Base is equal to the Base of the Segment, & its Side, to the Radius of the Sphere.

* By Sector of a Sphere is to be understood, a Portion containing a Segment, and a Right Cone on the Base of the Segment, whose Side is the Radius of the Sphere.

A N

APPENDIX;

ON THE THEORY

OF MENSURATION.

MENSURATION OF SUPERFICIES.

THE whole Theory of Menfuration of Superficies, confifts in finding a Rectangle equal to the given Figure.

In Theo. 17, Book 1ft. it is demonftrated, that every Triangle is equal to half a Parallelogram, on the fame Bafe and Altitude; and, in Theo. 18th. is fhewn and demonftrated, that, all Parallelograms or Triangles, having the fame or equal Bafes and the fame Altitude, are equal. Confequently, fince Rectangles are Parallelograms (Def. 33 and 44) and it is, there, fully demonftrated that they are all equal, having equal Bafes (i. e. having any two Sides equal) and the perpendicular diftance between thofe Sides and the oppofite alfo equal, it is evident, that the meafure of one is alfo the meafure of the other. And, becaufe Triangles on equal Bafes with Parallelograms, and being of equal height are equal half fuch Parallelograms, the Area of a Triangle is readily obtained.

Hence is deduced the general Rule (well known to all Surveyors or Artificers, concerned in meafuring fuperficies) for finding the Area of any triangular plane Figure; which is, to multiply the Bafe, i. e. any one Side, by half its height, from that Side, or half its Bafe by the whole height; the reafon for which is, in thofe Theorems, clearly accounted for; and, in that confifts the whole of fuperficial Menfuration.

In Prob. 20th is ſhewn how to conſtruct a Rectangle, equal to a Triangle, on thoſe Principles; and in the 22nd it is further extended to a Trapezium; or to any Quadrilateral, whatever, regular or irregular.

The 21ſt ſhews how to conſtruct a Parallelogram, under any Angle, equal to a Rectangle, on the ſame Baſe; and conſequently, by changing the Premiſes, a Rectangle may be formed equal to any Parallelogram whatever.

If the conſtruction of thoſe Figures be well underſtood; practical menſuration is eaſily attained; which conſiſts (as I have obſerved above) in finding a Rectangle, i. e. in knowing how to take the dimenſions of the two Sides of a Rectangle, equal to any given Figure; for, the multiplication of any two Numbers being applied to meaſure, always denotes, or produces the Area of, a Rectangle under ſuch dimenſions; which I ſhall, in the firſt place, endeavour to make clear and intelligible.

Moſt People, at the firſt thought on theſe matters, imagine, that if two Figures (of any ſpecies whatever) have equal Circuit, i. e. if the meaſure of all the ſides, of each Figure, in one ſum, be the ſame, they have equal Areas; than which, nothing is more falſe, as will be made appear.

A Circle contains the greateſt Area of any Plane Figure, having an equal Circumference or Perimeter. If a Circle be depreſſed, though ever ſo little, it becomes elliptical; in which Caſe, it loſes of its dimenſions; and the more it is depreſſed, i. e. the more excentric an Ellipſis, having an equal Periphery, or Circumference with a Circle, the leſs is its Area to that of the Circle. Conſequently, if a cylindrical Veſſel be bulged, or it its Sides are depreſſed till it becomes elliptical, it loſes of its meaſure; and the more the Sides are depreſſed, the more it loſes; becauſe it is evident, that, if they are preſſed quite flat, it loſes the whole internal Area.

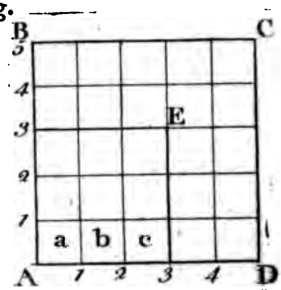
So likewiſe, of right lined Figures; the more the Sides are multiplied, and the nearer it approaches, in figure, to a Circle, the greater area it contains. Conſequently, regular Poligons, of a greater number of Sides, contain a greater Area, than thoſe of fewer Sides and having equal Perimeters.

Hence

Hence it is plain, that a Rectangle contains a greater Area than any other Quadrilateral whatever; if the measure of their Sides, in one Sum, be equal. But, a Square, which is the most perfect Rectangle, contains a less Area than a regular Pentagon; and a Pentagon contains less than a Hexagon; a Hexagon less than a Heptagon, and so on to a Circle, whose Perimeters are all equal; so that the Square and Circle, whose Perimeter and Circumference are equal, differ greatly in Area, as shall be illustrated hereafter to demonstration.

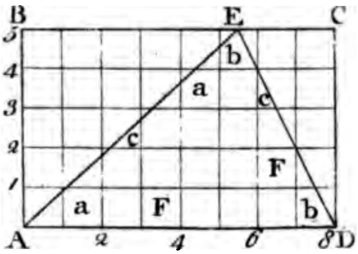
Now since, in Mensuration, a Rectangle is the standard or criterion, by which the Area of all Plane Figures, as well as all other superficial Contents, are ascertained; I shall, in the first place, shew, and account for, the methods of taking the dimensions of various Figures; which measures, being multiplied into each other, will give the true Area; each Figure being equal to a Rectangle under those Dimensions.

1. The area of a Square is obtained, by multiplying the measure of one Side into itself. e. g. ——



Let the Side of the Square ABCD be five feet; the measure of a Foot being represented by each small division on its Side, as 1, 2, 3, 4, 5. Now, if from these divisions, lines are drawn, both ways, parallel to the Sides; it will be divided into as many less Squares, as the Side multiplied by itself produces; viz. 5 times 5, equal 25; which, is the number of small Squares contained in the large Square, ABCD; each small square a, b, c, &c. being supposed one Foot, on each Side, is, therefore, called a square Foot. So, the Square AE, containing three feet on each Side, is a Square Yard; its Area is nine square Feet.

But, when a Rectangle measures more, or is longer, on one Side than the other; then, either Side multiplied by the other, not opposite, gives its Area or superficial Measure. Whereas, in a Square, the Sides are all equal; and consequently, the measure of one is also the measure of any other; therefore, the Side is said to be multiplied into itself, to produce its Area.



2. If the Rectangle ABCD meaſures 8 Feet on one Side (AD) and 5 Feet on the other (AB) the oppoſite Sides are the ſame (15. 1.) and if Lines are drawn through the diviſions (as in the Square) parallel to the Sides, it will be divided into 5 times 8, equal 40; the number of ſquare Feet, contained in the Rectangle.

Hence, it is evident, that, in Menſuration, any two Numbers denoting Meaſure, in Inches, Feet, Yards, &c. being multiplied into each other, gives the area of a Rectangle under theſe Dimenſions, in ſquare Inches, Feet, &c. and conſequently, all multiplication of Meaſure, denotes the Figure, under ſuch dimenſions, to be right angled. Wherefore, the multiplication of Lines, in Geometry, implies a Rectangle conſtructed on two Lines.

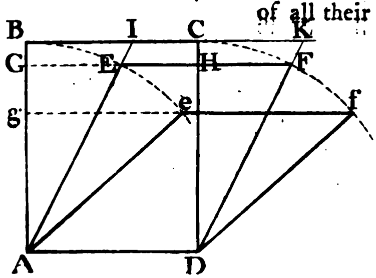
3. To any Point, E (in BC) let AE, DE be drawn, forming a Triangle, AED.

It is obvious, that the Triangle AED contains ſeveral whole and entire Squares, F, F, &c. and ſome Pentagons, as a, a; ſome Trapezia, as bb, and ſome Triangles, as cc, &c.

Now, it would be no eaſy matter to aſcertain how many entire Squares all thoſe irregular Figures are equal to; for there are but 12, entire, 4 Pentagons, 3 Trapezia, and 7 Triangles.

But, ſince (by Th. 17) we have full conviction, that the Triangle AED is equal to half the Rectangle ABCD; and the Rectangle ABCD contains 40 ſmall Squares; conſequently, all thoſe irregular Figures are equal to 8 Squares; which, added to the 12 entire ones, make 20, the true Area of the Triangle AED.

4. Next, I will ſhew, that two Parallelograms, may have the Sum of all their Sides equal, and differ greatly in Area.



The Rect. ABCD, and Par. AEFD, have their Sides, AD, BC, and EF, equal; for, AD is common to both; conſequently, BC and EF, being both equal to AD (15. 1.) are equal to each other - Ax. 3. AB=AE, and DC=DF (N.B. 1, Def. 20.) and conſ. AB+BC+CD+AD is equal to AE+EF+FD+AD.

But, the Area of the Par. AEFD is not equal to the Area of the Rect. ABCD. For, if AE be multiplied into AD, it will produce an Area equal to the Rect. ABCD, becaufe, AB=AE.

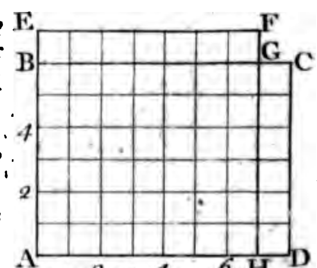
But, the Par. AEFD is equal to the Rect. AGHD only. For, it is demonftrable, that, the Triangle AGE=DHF. - 7. 1. Confequently, if DHF be taken away, and its equal, AGE, be added, the Rectangle AGHD is equal to AEFD.

Again, if AE be produced to I, and DF to K (in BC produced) Then, the Parallelogram AIKD is equal to the Rect. ABCD. For, the Triangle ABI=DCK; wherefore, taking away DCK, and adding an equal, ABI, the thing is manifeft.

Now, if the Rectangle ABCD be fuppofed to be depreffed, to AEFD, it is evident that it has loft of its dimenfions, confiderably; and if it be depreffed lower, to e f, it ftill lofes more; notwithftanding the Perimeter, A e f D, remains the fame, equal ABCD. Confequently, ABCD, contains a greater Area than any other Parallelogram, having an equal Bafe, and equal Perimeter.

Therefore, if ABCD be depreffed, i. e. if the Angles are not Right ones, it will contain a lefs Area; for, if A B and CD deviate ever fo little from a Perpendicular, BC muft neceffarily fall lower. Confequently, its Altitude being lefs, and the Bafe remaining the fame, its Area is lefs. - - - Cor. 1. 6.

5. It may feem ftrange, to fome, that a Square fhould contain a greater Area than any other Rectangle, having equal Perimeters.



Suppofe the Rectangle ABCD to meafure, on one Side (AD) 8 feet, and on the other Side (AB) 6 feet; its Area is $8 \times 6 = 48$; and its Perimeter is $8+8+6+6=28$.

Let a Square, AEFH, be defcribed on AH, equal 7 feet. Then, the meafure of its Sides, in one fum, is 7×4, or 4 times $7=28$, the fame as the Rectangle.

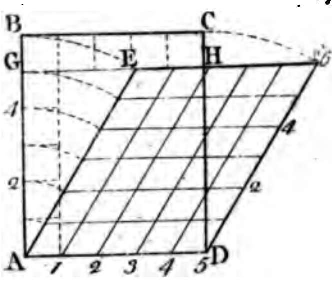
Now, the Rectangle ABGH is common to both the Square, AEFH, and the Rectangle ABCD.

But the Rectangle, BEFG, which is the remaining part of the Square, contains 7 fmall fquares; whereas, the Rectangle DCGH, the remaining part of the Rectangle, ABCD, contains but fix.

 The

The area of the Square is 7×7; or 7 times $7=49$, whe[re]as the area of the Rectangle is but $8 \times 6=48$; one 49th part leſs, in its Area, than the Square; which is accounted for by the Figure; and from the 11th Theorem, Book 5th.

Hence, it is plain, that a Square is the moſt perfect of Rectangles, or other Parallelograms; as a Circle, of all Plane Figures. And, ſince no other Figure, but Rectangles, can anſwer the purpoſes of Menſuration, in producing the true Area; it is certainly moſt conſiſtent, that all ſuperficial Meaſure ſhould be reduced to a ſtandard meaſure by the moſt perfect of Rectangles, a SQUARE.



6. In the Parallelogram AEFD, let AE be divided into the ſame equal parts as AB or AD; and if, through thoſe diviſions, Lines are drawn parallel to the Sides, AD and AE, it will be divided into as many leſſer Parallelograms as the Rectangle, ABCD, contains Squares.

But, thoſe leſſer Parallelograms are Rhombuſes (Def. 36) each being in the ſame Ratio to a Square, of an equal Perimeter, as the whole Parallelogram AEFD to the Rectangle ABCD; conſequently, they do not aſcertain its Area.

The Area, therefore, of a Parallelogram (except it be a Rectangle) is not obtained by the meaſure of its two Sides, AD, AE, or DF, but by the meaſure of one Side, as AD, only, and its perpendicular Altitude, DH; which, being multiplied together, produce the Area of a Rect. AGHD; to which, the Par. AEFD, is equal; as it is evident from the Figure.

The Par. AEFD=AGHD, and is deficient of the Rect. ABCD by the Rectangle GBCH, equal one ſixth part of ABCD (1. 6) GB being one ſixth part of AB.

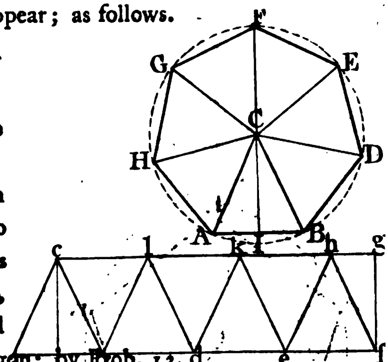
N. B. Either Side of a Parallelogram may be taken, in order to obtain its Area, but not both; unleſs it be a Rectangle.

If DF be taken, then, the perpendicular diſtance between that Side and its oppoſite, AE, will alſo give the true Area.

7. The

7. The Area of regular Poligons is readily obtained.

The general Rule for measuring Poligons is, to multiply half the Perimeter, i. e. half the Sum of all the Sides, by a Perpendicular from the Center to any Side; which implies, that a Rectangle of those Dimensions is equal to the Poligon; the truth of which I shall endeavour to make appear; as follows.



Let ABDEFGH be a regular Heptagon, whose Center is C.

Draw AC, BC, &c. dividing it into seven equal, Isosceles Triangles.

Draw a Right Line, a f, indefinite; in which, take a b, b d, &c. each equal to a Side of the Poligon; and, on the Bases a b, b d, &c. construct the Triangles, a c b, b l d, d k e, and e h f, equal and similar to the Triangle ACB in the Poligon; by Prob. 12.

And, if a Right Line, c h, be drawn through their Vertices, it will be parallel to a f (C. 18. 1.) and, there will be constructed three more Triangles c b l, &c. equal and similar to the other; and consequently, the Trapezium, a c h f, is equal to the Heptagon AGEB: for, it contains seven Triangles, equal to those in the Poligon.

Draw c i perpendicular to a b; the Triangle a c b being Isosceles, the Base, a b, and consequently, the Triangle, a c b, is bisected, by the Perpendicular c i. - - - C. 3. 9. 1.

Produce c h; and, make h g equal a i and draw f g, which will be parallel to c i, and also equal. - - - 15. 1.
wherefore, the Tri. f g h, is equal to the Tri. a c i. - 7. 1.
consequently, the Rectangle i c g f is equal to the Trap. a c h f; (which was made equal to the Poligon.) - - Ax. 6 and 7.

But, the Side i f, equal c g, of the Rect. i c g f, is equal to IBDEF, half the Perimeter of the Poligon; and c i, equal g f, the other Side of the Rectangle, is equal to C I, a Perpendicular of the Poligon; by Construction.

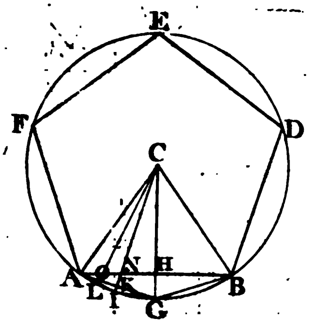
Consequently, half the Sum of the Sides, AB, BD, &c. (equal i f) multiplied by the Perpendicular CI (equal c i) gives the Area of the Heptagon ABDEFGH.

Which, Rule, holds good for all regular Poligons whatever.

8. From

8. From what has been advanced, concerning Poligons, we do with certainty infer, that a Triangle whose Base is equal to the Circumference of a Circle, and its Altitude equal to the Radius, is equal to that Circle; and consequently, half the Circumference multiplied by the Radius gives the Area of a Circle; for it is equal to a Rectangle, under the semi-Circumference and the Radius.

Also, every Poligon, circumscribing a Circle, is equal to a Triangle, whose Base is equal to the measure of all the Sides, in one Sum, and its Altitude equal to the Radius of the inscribed Circle.



The Pentagon ABDEF, it is evident, is less, in its Area, than the Circumscribing Circle; by the five Segments AGB, BD, &c.

Bisect the Ark AGB and draw CG, which will be perpendicular to AB; and draw the Chords AG, GB, which are the Sides of a Decagon, inscribed in the same Circle; and, if the Side AG be bisected, and CK drawn, it will be perpendicular to AG. - Cor. 3. 9. 1.

The two Sides AG, GB, of the Decagon, are greater than the Side AB, of the Pentagon (13. 1.) and, the Perpendicular CK, of the Decagon, is greater than the Perpendicular CH, of the Pentagon; for, CN is longer than CH - - - C. 2. 12. 1.

And CK is longer than CN; consequently, the Area of the Decagon, whose Perimeter is ten times AG or GB, and its Perpendicular is CK, must be greater than the Pentagon, whose Perimeter is five times AB, and its Perpendicular is CH.

The difference is five Isosceles Triangles, equal ABG; whose Base is AB, the Side of the Pentagon, and the equal Legs or Sides, AG, and GB, are the Sides of the Decagon.

Again, the Area of the Decagon is less than the Circle, by th ten Segments AIG, GB, &c.

CK, produced, bisects the Ark AIG at I; draw the Cho AI and IG; which are the Sides of a Poligon of twenty Si whose Perpendicular is CO.

This Poligon is greater than the Decagon, by ten Isosceles Triangles, equal AGI, in the Segment AG or GB; and it is still less than the Circle by twenty Segments AI or IG.

9. From all which, it is clear, that every Poligon, inscribed in a Circle, is less than the Circle, by as many Segments as the Poligon has Sides; and consequently, every Poligon, circum scribed, is greater than the Circle.

But, the greater the number of Sides of the Poligon, the nearer it is to the Area of the Circle, i. e. to the Circle itself; and consequently, it must at last end in the Circle; that is, the Perimeter of the Poligon, will be equal to the Circumference of the Circle, and the Perpendicular (from the Center) to the Radius.

For, if AI be again bisected in L; the Sides AL, LI, of a Triangle, in the Segment AI, are greater than the Chord AI, which is the Base of that Triangle; yet, they are less than the Arks of the Segments AL, LI; but they must, at last, by multiplying them, end in the Circumference.

So likewise, the Perpendicular CH, of the Pentagon, is less than CK, the Perpendicular of the Decagon, which is also less than the Perpendicular CO, of the Vigintagon. And, if AL is again bisected, and a Perpendicular drawn to the Chord AL, it will be greater than the Perpendicular CO; and will at last be equal to the Radius of the Circle.

Wherefore, the Circle being conceived to be formed of an infinite number of Isosceles Triangles, whose common Vertex is the Center, and whose Bases are in the Circumference, and which are, altogether, equal to it; consequently, a Rectangle under the Semi-circumference and the radius of the Circle; or, under half the Perimeter of the innumerable sided Poligon (equal to the Circumference) and its Perpendicular (equal the Radius) is equal to the Circle.

And consequently, a Triangle whose Base is equal to the Circumference, and its Altitude equal to the Radius, is also equal to the Circle; for every Triangle is equal to half a Rectangle of the same Base and Altitude (17. 1.) Therefore the Triangle, is

 equal

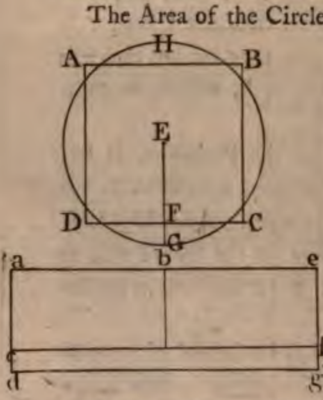
equal to a Rectangle under half its Base, and the Altitude. Hence, the Rule for meaſuring a Circle is, to multiply half the Circumference by the Radius.

From hence it is clear, that, of all Figures which have an equal Perimeter or Circumference; i. e. whoſe Bounds are equal to the ſame Right Line; a Circle contains the greateſt Area.

10. I have already ſhewn, that a Square contains a greater Area than any other Rectangle, having an equal Perimeter, or the meaſure of all their Sides equal. I will next ſhew, how much it is leſs than a Circle, having equal circuit.

Suppoſe the Sides of the Square, ABCD, be each equal to a fourth part of the Circumference of the Circle, GH; then the four Sides are equal to the whole Circumference.

The Area of the Circle is greater than the Area of the Square.



From what has been ſaid, concerning Poligons and Circles, it is evident; that the Circle, GH, is equal to a Rectangle, two oppoſite Sides of which, are equal to the Circumference; the other are each equal to the Radius EG.

But, the Square is equal to a Rectangle, two oppoſite Sides being equal, each to two ſides of the Square, i. e. equal half the circumference of the Circle; and the other two Sides, each equal to the Perpendicular, EF, only; which, being conſiderably leſs than the Radius EG; conſequently, a Rectangle, a e g d, whoſe Sides a e, d g, are each equal half the Circumference, GH (equal a e, c f, of the Rectangle a e f c) and the Side e g (equal a d) is equal the Radius, EG (but, the Sides a c, e f, each equal the Perpendicular EF, only) will have a greater Area than the Rectangle a e f c; and the difference is the Rectangle c f g d.

The Circumference of a Circle is to its Diameter, nearly, as 22 is to 7; which, is near enough, for meaſuring any Circle that can be formed. But more exact, the Diameter being 113, the Circumference will be 355. In both, the Circumference is too much; the laſt, by little more than one five millionth part of the Diameter.

Suppoſe then, by the laſt, the Diameter, GH, to be 17, the Circumference will be 53,4, very near. The Side of the Square, ABCD, being a fourth part of the Circumference, is 13,35, the Perpendicular, EF, half the Side, is 6,675; and the Radius of the Circle EG, is 8,5; the Area of each is as follows.

AB+BC=	26,7	half the Circumference	26,7
× by EF=	6,675	× by the Radius, EG, =	8,5
	1335		1335
	1869		2136
	1602		
	1602	Area of the	226,95 Circle
			178,2225
Area	178,2225 of the Square		
		difference	48,7275

A regular Pentagon, having the ſame Perimeter as the Square, is greater than the Square, but it is leſs than the Circle; alſo a Hexagon, is ſtill greater than a Pentagon, of equal Perimeter. A Heptagon, is ſtill greater than a Hexagon; and an Octagon, greater than a Heptagon; ſo that, the more the Sides are multiplied, the nearer it approaches to the Circle; for, the Perimeter of the Poligon, by multiplying, will end, or fall, at laſt, into the Circumference of the Circle.

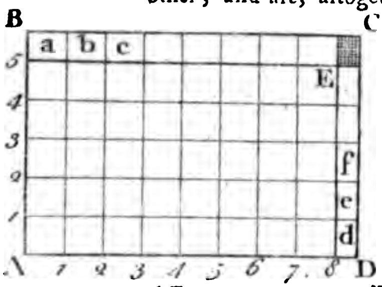
11. From what has been advanced, it is manifeſt that the whole theory of Menſuration, of Superficies, conſiſts in finding the ſides of a Rectangle, equal to the given Figure; ſince the Multiplication of any two meaſures (into each other, or one by the other) denotes a Rectangle of thoſe dimenſions.

Having ſhewn how the meaſures of Plane Figures, from a Square to a Circle, are taken, ſo as to produce their Areas; I ſhall, in the next place, account for the multiplication itſelf, by which, the true Area is aſcertained.

Suppoſe, then, a Rectangle ABCD, whoſe Sides, AB and CD, meaſure, by ſome Scale of equal Parts, 5 and 9 twelths, each; and the other Sides, AD and BC, each 8 and 6 twelths.

Through the Diviſions 1, 2, 3, 4, &c. on the Sides AB and AD, which may repreſent the meaſure of a Foot or 12 Inches, each (for, you may ſuppoſe any meaſure whatever, to be repreſented by them) if Lines be drawn, parallel to the Sides of the Rectangle, as in the Figure, it will be divided into as many whole

and entire Squares, as the whole numbers produce; viz. the Rectangle AE, whoſe Sides, A 8, A 5, being multiplied, 5 times 8, is equal to 40. The fractional parts a, b, c; d, e, f, &c. produce 8 more Integers or Squares, and 7 eighths of another; for, the 8 ſmall Rectangles a, b, c, &c. to E, are each 3 fourths of a Square; being the full meaſure on one Side, and but 9 twelths on the other; and are, altogether, equal to 6 entire Squares.



The 5 ſmaller Rectangles d, e, f, &c. to E, being but 6 twelths (half the Integer) on one ſide, are each equal to half a Square; and make, together, two and a half; and the ſmall Rectangle EC, being 3 fourths of one of theſe, or half one of the other, is equal 3 eighths of a Square.

AE =	40
BE =	6
DE =	2 - 6
CE =	0 - $4\frac{1}{2}$
ABCD	48 - $10\frac{1}{2}$

The large Rectangle, AE, is equal to 40 Squares; to which, if the Rectangles BE, DE, and EC be added; BE=6, DE=$2\frac{1}{2}$, and EC=$\frac{3}{8}$ of a Square; together, they are equal to 8, and 7 eighths; which, added to 40, makes 48 and 7 eighths; the ſum total of the Area of the Rectangle ABCD, in ſquare meaſure, by that Scale.

12. Next, I will ſhew how the Product ariſes, by multiplying Feet and Inches, together; commonly called Croſs Multiplication.

The meaſures of the two Sides of the Rectangle, AB and AD, 8 - 6, and 5 - 9, being placed one under the other ('tis not material which is firſt, or uppermoſt) as in the margin, we begin with the Feet, in the multiplier, and ſay, 5 times 6 is 30.

```
 8 - 6
 5 - 9
-------
42 - 6
 6 - 4 - 6
-----------
48 - 10 - 6
```

Now, it is evident, from the Figure, that this is Feet, one way, and Inches, the other; and is the Rectangle DE, 5 feet long, E 8, by 6 Inches, 8 D.

Feet multiplied by Inches, or Inches multiplied by Feet, produce Inches; i. e. the twelfth part of a ſquare Foot; 30 Inches is, therefore, equal to 2 feet and 6 Inches.

'Tis the ſame, if you begin with the 9 inches, firſt.

Set

Set down the 6 Inches, under Inches, and carry the two Feet forward to the whole Numbers, and ſay, 5 times 8 is 40; which being Feet both ways, they conſequently produce ſquare Feet, and is the large Rectangle, AE. The two Feet, ariſing from the Rectangle BE, bing added, make 42 Feet; which, ſet down under the Feet.

Next, take the Inches, in the multiplier, and ſay, 9 times 6 is 54.

Now, theſe are Inches both ways (the ſmall Rectangle EC) and produce only Parts, or ſquare Inches, i. e. the 144th part of a ſquare Foot. For if the Rectangle EC be divided, on each Side, into Inches (viz. 9 and 6) and lines be drawn through them, as in the large Rectangle AE, it will contain 54 ſquare Inches, which are called Parts; twelve of which are called an Inch; for, it muſt be obſerved, that what is here meant by an Inch, is not a ſquare Inch, as by a Foot is underſtood a ſquare Foot; but it is the 12th part of a Square Foot, equal 12 ſquare Inches, or Parts.

Wherefore, the ſmall Rectangle EC, which contains 54 ſquare Inches, it is evident, is equal to 4 Inches and 6 Parts only; ſince twelve of theſe Parts are but one Inch, and 4 times 12 is 48, there remains 6 Parts, or half an Inch; which, ſet down, one place to the right hand of Inches, and carry forward the Inches to the next place.

Laſtly, ſay 9 times 8 is 72, which is the Rectangle BE, 8 Feet one way (5 E) by 9 Inches the other (5 B) and produces Inches; to which, add the 4 Inches from the laſt place, it makes 76 Inches, equal 6 Feet 4 Inches; ſet them down, as above, Feet under Feet, and Inches under Inches; which, added into one Sum, makes 48 Feet, 10 Inches, and 6 Parts (equal half an Inch) the Area of the whole Rectangle ABCD, in ſquare Feet, Inches, and Parts.

Thus, multiplying Feet and Inches together, the Area of a Rectangle is obtained, whoſe Sides are equal to ſuch Dimenſions. 'Tis needleſs to proceed further; as it is evident, if Parts were taken, they would be accounted for, after the ſame manner.

13. As there are many Perſons, who meaſure accurately by Feet and Inches, yet are unacquainted with Decimals; I ſhall, briefly, explain the manner of multiplying Decimals; which is much eaſier, and better adapted to any Quantity whatever.

2 Multiplication,

Multiplication, by Feet and Inches, denotes the fraction of a Foot to be in twelfth parts, which are called Inches; and these are again divided into twelve, called parts, and so on to Seconds and Thirds; each of which is a twelfth part of the foregoing. Whereas, the Decimal Fraction implies the Integer, whatever it be, to be divided into tenths, hundredths, and thousandth Parts, and so on, by tens, *ad infinitum*, as the other by twelves, and are therefore called Duodecimals.

Multiplication, in Decimals, is performed in the same manner as whole Numbers; seeing, that the places increase regularly, by tens, from the Decimal to the Integer, as in whole Numbers.

Observe the following Example; by which, the Area of the Rectangle ABCD is produced, the same as by Feet and Inches.

```
 5,75
 8,5
------
 2875
4600
------
48,875
------
```

The measures of the Sides of the Rectangle, in Decimals, are as in the margin. AB, 5 feet 9 Inches, is, in Decimals, 5 Feet and 75 hundredth parts of a Foot, equal 9 Inches; and AD, 8 Feet and 6 Inches, is 8 feet, and 5 tenths of a Foot, equal 6 twelfths.

I place the least Side, AB, first, because it has more places of Figures, and would occasion another Line, if placed otherwise, the Product would be the same.

In the operation, the Figures are multiplied together, as in common Arithmetic, without any regard to the places of Decimals or Integers; and, when added together, as many places of Decimals as there are in both the Multiplicand and Multiplier, viz. three, are taken from the right hand of the Product; the remaining two are Integers, or Feet; which produces 48 Feet, and 875 thousandth parts of a Foot, equal 7 eighths of a Foot, equal 10 Inches and a half. For, 500 is half a Thousand, therefore equal to 6 Inches; and the remaining 425 is 3 eighth parts of a Thousand, therefore equal to 4 Inches and a half, which is 3 eighth parts of a Foot.

The value of a Decimal Fraction is easily obtained, as follows.

Let the Integer be whatever it may; multiply the Decimal by the number of parts of the next inferior denomination of the Integer; which, in this Case, is Inches, or the twelfth part of a square Foot, to which I would reduce the Decimal 875.

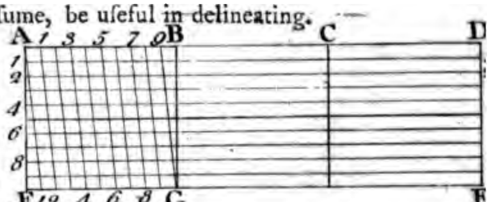
Multiply it by 12, and take away from the Product, as many places as you had at firſt, there is left 10 Inches; and the Decimal, 500, are now thouſandth parts of an Inch; conſequently, equal to half an Inch, or 6 Parts, as a ſecond operation evinces; 12 being, again, the next inferior Denomination, as twelve Parts are equal to one Inch.

875
12
——
10,500
12
——
6,000

If the original ſtandard meaſure of a Foot was, inſtead of Inches, divided into tenth parts of a Foot; or, which is the ſame thing, if an Inch had been made a tenth part of a Foot, inſtead of a twelfth part, and the Inches ſubdivided into tenths inſtead of eighth parts, Menſuration would have been greatly facilitated, by decimal Parts inſtead of Duodecimals, or multiplication of Feet and Inches. The meaſures being taken, at once, in Decimals (which, by a common Rule, is difficult to do, accurately, and loſs of time to reduce ſome meaſures to a Decimal) would be more uſeful in all Caſes whatever.

I am not a little ſurprized, that there has been no attempt to make it univerſal, as every Perſon, who has experienced it, muſt give the preference to Decimals, which is infinitely preferable to any other manner of dividing.

14. I ſhall, here, ſhew how to make a Scale, and divide it into Decimal Parts, which may, I preſume, be uſeful in delineating.



Let AB be a determined Meaſure repreſenting one Foot, or any other meaſure required.

Produce AB, and repeat the meaſure of AB, to C and D, as often as you pleaſe.

Draw AF and DE perpendicular to AD; on either of which, make, at pleaſure (i. e. with any opening of the Compaſſes) ten equal diviſions, as on AF; and, through them, draw lines parallel to AD, as 1, 1 and 2, 2, &c. in all, eleven lines.

Divide AB and FG into ten equal parts (Prob. 36) and draw A 1, 1 2, &c. diagonal-wiſe, as in the Figure, and proceed in the ſame manner, to 9 G, parallel to each other.

By

By which means, the length or meaſure given, AB, is divided into a hundred equal parts; each interval, from AF to A 1, the firſt Diagonal, at 1, 2, 3, &c. expreſſing all the Units from 1 to 10.

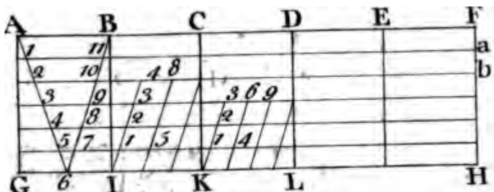
Between each of the Diagonals, as A 1, and 1 2, &c. the intervals are ten of thoſe hundredth parts, being every where equal to the Diviſions on AB.

In the ſame manner, any Meaſure whatever, either greater or leſs than AB, may be accurately divided into Decimal Parts.

After ſuch plain directions, in the Conſtruction, the application of it, in Practice, would be impertinent and unneceſſary.

15. In the ſame manner, a Scale may be divided into Duodecimals; by taking 12 Diviſions on AF, and dividing the given Meaſure, AB, into twelve, and proceeding as directed above.

Duodecimals is ſeldom practiſed, further than the diviſion of a Foot into Inches, which are the ſtandard Meaſure, on all common Rules; therefore, to make a Scale of Feet and Inches, to any determinate meaſure, obſerve the following directions.



Let AB be the determinate meaſure of a Foot; according to the proportion you intend to make a Drawing, or Deſign.

Produce AB, and repeat the meaſure of AB as often as you think neceſſary, as C, D, E, F.

Draw AG and FH, at right angles to AF, and make ſix equal diviſions on AG, at diſcretion; through which, draw 1 a and 2 b, &c. parallel to AF, and alſo BI and CK, &c. parallel to AG.

Biſect GI, or AB, and draw the Diagonals, A6 and B6; and the given Meaſure, AB, is divided into twelve equal parts, as they are numbered from the Perpendicular AG, on the Diagonals.

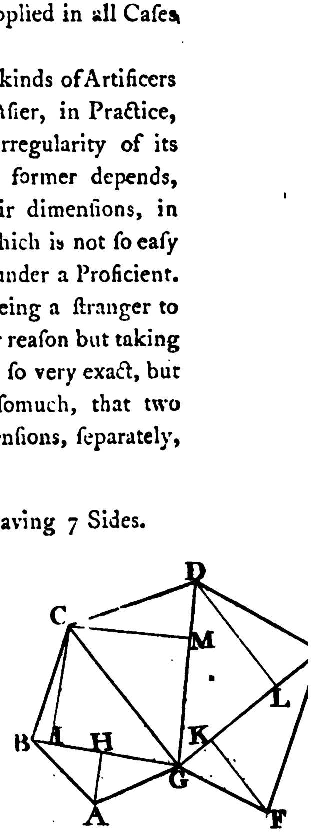
N. B. It may be done with only four, or three Spaces, as between IK and KL; but I think it more applicable, in Practice, with ſix; which is left to every Perſon's own diſcretion; the Diviſions are the ſame in them all.

Having

Having illuftrated and explained the whole rationale of menfuration of plane Superficies, and fhewn, how the Dimenfions of all regular Figures are taken, in order to find Rectangles equal to them; I fhall juft give one fpecimen of irregular Figures; for, I do not intend this as a Treatife on Menfuration, but only a brief Theory, to fhew how it is founded in Geometry; which, being well underftood and digefted, it is eafy to be applied in all Cafes whatever, which may occur, in Practice.

Menfuration of Superficies, applied to various kinds of Artificers Work, is ftill the fame in Theory, but much eafier, in Practice, than Surveying Land; on account of the irregularity of its Figure, and unevenefs of its Surface. The former depends, chiefly, on the mode, or cuftom of taking their dimenfions, in order to reduce the Work to fquare meafure; which is not fo eafy to defcribe, but requires very little application under a Proficient. For, a Perfon ever fo well verfed in Theory, being a ftranger to practice, would differ in his eftimate, for no other reafon but taking his dimenfions different; in which they are not fo very exact, but give and take all reafonable allowances; infomuch, that two Meafurers, of long practice, taking their dimenfions, feparately, would differ confiderably in their Sums total.

16. ABCDEFG is an irregular Heptagon; having 7 Sides.



Draw the Diagonals BG, CG, GD and GE, or otherwife, at difcretion, dividing the given Figure into Triangles, as ABG, BCG, &c. in each Triangle, draw Perpendiculars, as AH, CI, &c. The meafure of a Perpendicular and half the Bafe on which it falls, in Feet and Inches, or Decimal Parts, being multiplied into one another, gives the Area of each Triangle feparately; which, added into one Sum, is the Contents of the whole Figure (Ax. 2. 1.)

Or it may be done fomewhat readier, by taking any two contiguous Triangles, as ABG, BCG, which together, make a Trapezium, ABCG; unlefs it fhould happen to be a Parallelogram.

 The

The Diagonal, BG, is then a common Base to the two Triangles and the Area, of both, is obtained at once, by multiplying the Diagonal, BC, by half the Sum of the two Perpendiculars (Prob. 22.)

The Area of any other two contiguous Triangles, as GDE, GEF, forming a Trapezium, is had after the same manner; and if another Triangle remains, as GCD, its Area, being obtained, must be added to the Areas of the two Trapeziums, into one Sum; which is the true Area of the Heptagon ACDF.

In this Process it may be observed, that not one Side of the Figure is measured, but only the Diagonals and Perpendiculars, all which fall within the Figure. The reason is obvious; because the Sides are not at right angles with each other; or if some of them were, it would be of no use, unless they formed a Rectangle, or other Parallelogram, with the adjacent Diagonal, which is seldom the Case. Whereas, the Perpendiculars making Right Angles with the Diagonals, it is easy to account for the true Area being obtained by them; from 17 and 18th of the 1st of Elements.

N. B, If the Figure to be measured be a large Field, the measures are taken in Chains and Links, &c.

As it must be obvious to every one, who has gone thus far, that to reduce any irregular Figure to a Triangle, having an equal Area to the Figure, is the principal business of Mensuration of Superficies; I would advise the learner to be clear in the Construction, and also in the Demonstrations, of the 20, 21, 22, and 24th Problems. In respect of the 26th and 27th, I shall add further; that not only a Triangle, or Trapezium, but almost any right lined Figure may be divided into two equal parts by a Right Line, from any determined Point, in any Side; as follows.

17. ABCDEF is a Hexagon, having an internal Angle, at D.

Draw the Diagonal AD, dividing it into two Trapeziums.

Bisect AD, and divide each Trapezium, AC and AE, into two equal parts, by the Right Lines GH and GI, (Prob. 27.)

The whole Figure is divided into equal parts; HAI = CGE, Draw HI, and GB parallel to it; then, a Right Line, BI, divides the whole Figure equally.*

Now,

Now, B I biſects the hexagonal Figure A C F; but, the Points B and I are accidentally obtained.

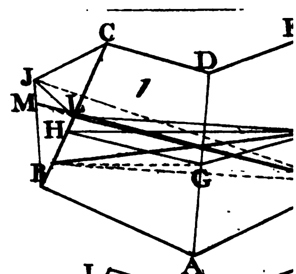
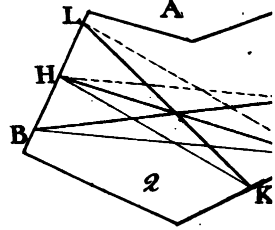


Fig. 2. Let K be the given Point, from which a Right Line is required to be drawn, biſecting the given Figure.

Draw BF, and IH parallel to it, and draw FH; which alſo biſects the Figure.

Draw KH, and FL parallel to it, and join KL. I ſay, KL divides the whole Figure equally in two.



The proof is in the 18th of the 1ſt of Elements, the ſame as in the 27th Problem; for, the Triangle B H F = B I F, and H L K = H F K.

After the ſame manner it may be done from any Point whatever, in any Side.

If the Figure had more Angles, as B J C (Fig. 1.); a Right Line may be drawn, after the ſame manner as B I, dividing the whole Figure equally; having perhaps two or three more operations. For, by biſecting B C, and drawing J L, the Point, where B I cuts BC, may, by one more operation, be brought to L (as KL) and then by joining K J and drawing LM parallel, KM, will biſect the whole Figure.

18. Now, ſeeing that all irregular Figures are reduced, to Triangles, in order to obtain their Areas; I will ſhew how the Area of any right-lined Triangle may be obtained without having a Perpendicular, by the meaſures of its Sides only.

In ſurveying Land, it may ſo happen, that, on account of Bogs, Water, Trees, &c. it would be very difficult, if not impoſſible, to obtain a Perpendicular, within the Figure, when its Sides are acceſſible. In ſuch Caſe, the Perpendicular may be found arithmetically, deduced from the 13th of the 2nd Book; or, the Area may be had without the Perpendicular, by the following Rule.

* Let it be obſerved, that if GB had fallen on this Side of the Angle (which is very probable) it would then be neceſſary to have recourſe to the third operation (Fig. 3.) in the 27th Problem; by which, the Line, BI, biſecting the Figure, would be obtained on this Side of the Angle B.

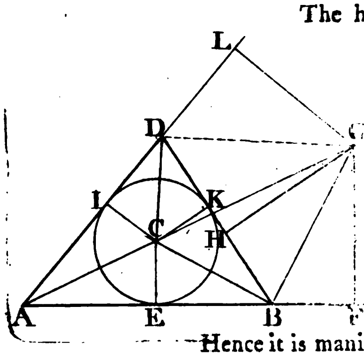
 From

From half the Sum of all the Sides of the Triangle, ſubtract each Side ſeverally; multiply the half Sum by the three Differences, continually; out of which Product, extract the Square Root; which is the Area of the Triangle, e. g.

Let the Sides of the Triangle ADB, be, AB–9, 8; AD–8, 5; and BD–6, 7. The half of which Sum is 12, 5; and the Differences, ſubtracting each Side, ſeverally, are, A B–2, 7; AD–4; and BD–5, 8. See the whole Operation, as follows.

AB—9,8	12,5	783 (27,98
AD—8,5	2,7	4 Area.
BD—6,7	875	47) 383
25,0	250	329
12,5 half the Sum of the Sides.	33,75	549) 5400
	4,	4941
Differences. { 2,7		
4,0	135,	5588) 45900
5,8 }	5,8	44704
	1080	1196
	675	

The half Sum, 783, multiplied into the Differences.



In the Triangle ADB, biſect every Angle by Right Lines, meeting at C; which is the Center of a Circle inſcribed. - - 4. 4.

The Triangle ABD is thus divided into three Triangles ACB, BCD and ACD; the Perpendiculars of all which, CE, CI, and CK, are equal; and conſequently, AE, is equal to AI; BE = BK, and DI = DK - C. 2. 16. 3.

Hence it is manifeſt, that AE × CE is equal to the Trap. AECI. (For, CE and CI are, reſpectively, perpendicular to AB and AD) Alſo, B E × C E = Trapezium B E C K, and D I × C I = D I C K. Produce A B; make B F equal to D I, equal D K.

Conſ. A F × C E is equal to them all, i. e. to the Triangle ABD.

I ſay, the Square Root, of AF × AE × EB × BF, is alſo equal to the Triangle.

DEM.

DEM. Because $AE=AI$; $BE=BK$; and BF equal DK - Con. AF is equal to half the sum of the Sides, AB, BD, and AD. Also, AE, EB, and BF, are respectively, the Differences of those Sides, subtracted from the half of their Sum; (for, $AE=AF-BD$; $EB=AF-AD$; and, $BF=AF-AB$.) Draw FG perpendicular to AF, produce AC to G, and draw GB. GB will bisect the external Angle, DBF, of the Triangle.

Draw GH perp. to BD; the Trap. FGHB is similar to EBKC. For, the Angles at F and H, E and K are Right; by Construction; wh. the Angle $FBH+FGH=$ to two Right Angles - Th. 1. 10. 1. and, $FBH+HBE$ are equal to two Right Angles - - 1. 1. conf. $ECK=FBH$ (for, $HBE+ECK=2$ Right Angles) Ax. 7. 1.

But, BC bisects the Angle ECK, by Construction; & $CE=CK$ wherefore, BG bisects the Angle FBH; and BH is equal to BF; and, consequently, the Triangles CBE, BGF are similar - 13. 6. Also, because FG is parallel to CE, the Triangles ACE, AGF are similar; (2. 6.) and therefore, $CE:AE::FG:AF$ - 4. 6. Also, because BCE, BGF are similar, $CE:EB::BF:FG$. and consequently, $CE\times CE\times AF\times FG=AE\times EB\times BF\times FG$.

Now, because FG is on both Sides, let it be rejected, and substitute AF, on both Sides; and consequently, it will be $CE\times CE\times AF\times AF$, i. e. $CE\,\square\times AF\,\square$ is equal to $AE\times EB\times BF\times AF$; i. e. to AF, the half Sum of the Sides, multiplied into the three Differences AE, EB, and BF.

But, AF multiplied by CE is equal to the Area of the Triangle; conf. it is the square Root of $CE\,\square\times AF\,\square$; which is equal to AF, multiplied into the three Differences. Q. E. D.

Mensuration of curved Surfaces, convex or concave.

18. The Areas of the Superficies of regular curved Surfaces, are easily deduced from the 10th, 11th, 12th, the 15th and 16th Theorems, Book the 8th.

By the 10th, the curved Superficies of a Cylinder is equal to a Circle, whose Radius is a mean Proportional between its Side and Diameter.

Or, it is obvious, from the 5th, that it is equal to a Rectangle, whose Sides are its Circumference and Height.

19. By

19. By the 11th it appears, that the conical Superficies of a Right Cone, is equal to a Circle, whose Radius is a mean Proportional between its Side and the Radius of its Base.

Or, it is equal to a Triangle, whose Base is equal to the Circumference of the Base of the Cone, and its Altitude, to the Side of the Cone.

20. By the 12th it is manifest, that the conical Superficies, of the Frustum of any Right Cone, is equal to a Circle, whose Radius is a mean Proportional between the Side of the Frustum, and the Radius of the Base, added to the Radius of the opposite Circle.

And every Circle is equal to a Triangle, whose Base is equal to its Circumference, and its Altitude to the Radius. Art. 8th.

Or, the conical Superficies is equal to a Rectangle, whose Sides are equal to the Side of the Frustum, and half the Sum of the Circumference of the Base, added to half the Circumference of the opposite Circle.

21. The Area of the Surface of a Sphere is equal to a Circle, whose Radius is equal to the Diameter of the Sphere. - 15.8.

And the Area of any spherical Segment is equal to a Rectangle, one Side of which is equal to the Circumference of the Sphere, the other to the height of the Segment.

For, it is equal to a cylindrical Surface of those Dimensions - 16.8.

The same holds true of any portion of the Surface of a Sphere, intercepted between parallel Planes.

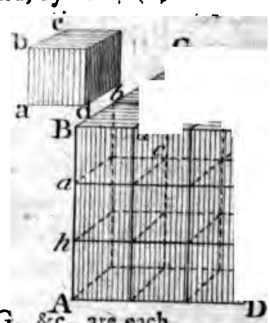
The spherical Surface, intercepted between parallel Planes, is therefore equal to a Rectangle, whose Sides are the Circumference of a large Circle of the Sphere, and the perpendicular distance between the parallel Circles.

These Rules, respecting curved Surfaces, being clearly understood, will be found extremely useful to Artificers, in measuring circular Halls, or Rotundas of any kind; and Domes, entire; or where a part is deficient by means of a Lanthorn, or otherwise, at the Top; the Surface between any two parallel Circles being determined by the last.

MENSURATION OF SOLIDS.

AS, in Menfuration of Superficies, the whole Bufinefs is to find a Rectangle equal to the Figure propofed; and to determine how many Squares of a certain Dimenfion, the Figure is equal to; fo, Menfuration of Solids confifts in determining how many cubical Feet, &c. are contained in the propofed Solid.

In order to which, it is neceffary to know the affinity and proportion between one Solid and another, as contained in the 7th and 8th Books of thefe Elements; and particularly Parallelopipeds, to which all other Solids muft neceffarily be reduced, in Menfuration; a right angled Parallelopiped being of the fame importance, in refpect of Solids, as the Rectangle amongft Plane Figures. Alfo, as a Square is the Criterion of fuperficial meafure, fo a Cube, being the moft perfect Parallelopiped, is the ftandard, by which the Quantities of Solids are compared and eftimated.



For, the Definition of a Cube, fee Def. 10, 7.

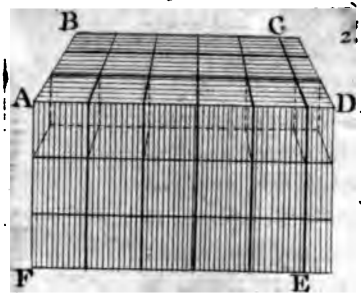
1. Let AGE be a Cube, whofe Sides, AB, &c. are each equal to 3 feet; alfo, let a c d be a Cube, whofe Side is one foot, or inch, &c. Now, fuppofe the Cube a c d to be the Unit of meafure; it is required to know, how oft the Unit, a c d, is contained in A G E.

It is evident, feeing, the Meafure of AB, BG, &c. are each equal to three times ab, b c, &c. that the Face A B C D, contains the Square b d, 9 times; every other Face, B C F G, or C D E F, the fame; therefore, each fmall Cube *a b c*, *d e f*, *e f g*, &c. made by Sections of Planes through *a g*, *g l*, &c. as in the Figure, are equal to a c d. Wherefore, fince each Surface contains 9 Squares, as BF; and if B *a*, C *g*, &c. be one foot in thicknefs, confequently, the Solid *a* G *l*, contains the fmall Cube, a c d, 9 times. But, AB is equal to three times a b; wherefore, the whole Cube A G E contains the fmall one 27 times.

For, if it be fuppofed to be cut, by parallel Planes, through *a g l* and *b i k*, parallel to the Top and Bafe, the Parts *a* G *l*, *a i l*, and *b* D *k* are equal, feeing that their Surfaces are equal;

and

and being also of equal thickness, C*g*, *g i*, *i* D; consequently, B *g* F, *a i l*, and *b* D *k* are equal (15. 7.) But, B *g* F G is equal to 9 times a c b; therefore, AGE is equal to three times 9, =27.



2. It is manifest, that if the Solid A C E (being a Parallelopiped) was longer, equal 5 times a b, having equal thickness, it would contain the small Cube, a c b, 45 times, i.e. five times 9; and thus it will increase, as often as the measure a b is added in length.

Suppose half a b be added; it will contain half 9 Cubes, i. e. $4\frac{1}{2}$; seeing that the Surface of the End, or Section through C E, contains 9 Squares, and the Solid C E has but half a Cube, i. e. half a b in thickness. Therefore, the whole Solid A C E contains the Cube a c b 49 times and a half.

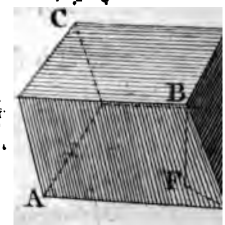
Hence, the measure of a right-angled Parallelopiped is to multiply the Side AD by AB, which gives the Area of one Surface (the Top, or Base) and, that Product being multiplied by its height or thickness, AF, i. e. by the number of times it contains a b (as, in this Case, three times) the Product of that multiplication is the Solid Contents of the Parallelopiped A F H.

But, all Parallelopipeds, having equal Bases and Altitudes, are equal (17. 7.) Therefore, whether it be right or acute angled, having obtained the Area of any one Face, in square measure (Art. 12.) that Product being multiplied by the perpendicular Distance, between that Face and its opposite, gives the solid Contents of the Parallelopiped. e. g.

3. Let A B C D be an acute angled Parallelopiped whose Base A D is 7 Feet, 9 Inches, on one Side, AE; the other, ED, 5 Feet, 4 Inches, its Area in square measure, is 41 Feet, 4 Inches.

7-9
5-4
38-9
2-7
41-4
3-8
124-0
27-6-8
151-6-8

The height of the Parallelopiped, BF, being 3 Feet, 8 Inches, it is obvious, that the whole Solid contains 3 times 41-4, and $\frac{2}{3}$; i. e. as often as the measure of a Foot is contained in the height, BF, so often the Solid contains cubical Feet, as its Base contains square Feet.



But, the Perpendicular BF, is 3 Feet and 8 Inches; consequently, the Solid will contain 41-4, 3 times and two thirds, (8 Inches, being two thirds of a Foot.) Wherefore, 41 Feet, 4 Inches multiplied 3 times, and two thirds, equal 151 Feet, 9 Inches, and 8 Parts, is the true Area, or solid Contents of the Parallelopd. ABCD.

Note. What is here meant by an Inch, in solid measure, is the twelfth part of a Cube Foot; viz. a Foot square, and one Inch in thickness; or, that Quantity in any other Figure. And the Part, in Solid measure, is 12 cubical Inches; or 12 Inches in length, and one Inch in breadth and thickness: a cubical Inch is, consequently, a Second of a cubical Foot.

It is obvious, that the Side BG, being inclined to the Base AD, is longer, than the Perpendicular BF, and consequently cannot give its true Area, for (joining GF) BFG is a right angled Triangle, of which BG is the Hypothenuse, therefore it is longer than BF (12. 1.)

Hence it is manifest, that all Parallelopipeds, or Prisms whatever, (for Parallelopipeds are Prisms) having equal height, have that Proportion to each other, which is between their Bases. And, having equal Bases, they are consequently, as their Altitudes.

The Rule, therefore, for measuring any Prism whatever, is to multiply the superficial Area of its Base, by its perpendicular height.

And consequently, the same Rule is applicable for Cylinders - 5.8.

4. Every Pyramid is equal to the third part of a Prism having equal Bases and equal Altitudes. - - - - Th. 4. 8.

Also, every Cone is equal to the third part of a Cylinder - 5.8.

And consequently, Pyramids and Cones, having equal Bases and Altitudes, are equal.

Hence the Contents of Pyramids and Cones are obtained; viz. by multiplying the Areas of their Bases by one third of their height.

Or, if multiplied by the whole height, it gives the Contents of a Prism or Cylinder, of equal height; consequently, a Pyramid, or Cone, is one third part of such a Prism, or Cylinder.

5. To find the ſolid Contents of a Sphere.

Having obtained its Diameter; find the Area of a Circle of that Diameter (Art. 8.) which being multiplied, by the Diameter, gives the Contents of a Cylinder, whoſe height and Diameter are equal.

The Sphere is equal to two thirds of ſuch a Cylinder-Th. 9. 8.

Or, having obtained the Area of a large Circle, of the Diameter of the Sphere; multiply the Product by two thirds of the Diameter.

Otherwiſe. Every Sphere is equal to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius - 17.8.

Therefore, having obtained the Area of its Surface, multiply that Area by one ſixth part of the Diameter.

See an Example, in each, in the Margin.

Let the Diameter of a Sphere be 15; the Circumference of a Circle, of that Diameter, is 47,14, nearly; by the Ratio of 7 to 22.

	Then, by Art. 8. the Semicircumference (23, 57) being mul-
23,57	tiplied by the Radius (7,5, half 15) gives the Area
7,5	of a great Circle of that Sphere, equal 176,775;
———	which multiplied by 10 (two thirds of the Diameter)
117 85	gives 1767,75, or three fourths of the Integer.
16499	Multiplying by 10, is only giving one place more of
———	Integers; ſeeing that, a Cypher, being added, makes
176,775	no difference in the Decimal; 750 thouſandth parts,
———	being equal to 75 hundreth parts; i. e. equal to
1767,75	3 fourths.

Secondly. 15 being the Diameter of the Sphere, find the Area of its Surface (Art. 21.) which is equal to a Circle whoſe Radius is equal to the Diameter of the Sphere.

The Circumference of every Circle has the ſame Ratio to its Diameter (Cor. 1. 14. 6. El.) conſequently, the Circumference of a Circle, whoſe Area is equal to the Surface of the Sphere, is double the Circumference of the Sphere; half of which, is equal to the whole of the other, equal 47,14.

Multiply

Multiply 47,14 by 15, the Diameter of the Sphere, the Product is the Area of the Surface of the Sphere.

Now, ſince the Contents of the Sphere is equal to a Cone, the Area of whoſe Baſe is 707,1; and its Altitude equal to its Radius, viz. 7,5; and a Cone is equal to one third of a Cylinder of the ſame Baſe and Altitude, therefore, 707,1, multiplied by 2,5, one third of 7,5, will produce the ſame Contents, as before, viz. 1767,75; as in the Margin.

```
  47,14
  15,
 -------
  707,1
    2,5
 -------
  35355
 14142
 -------
1767,75
```

6. The ſolid Contents of a Segment of a Sphere is obtained from the 17th of the 8th, Cor. 3rd.

The ſolid Contents of the Sphere being equal to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius, as above; and conſequently, the Contents of any Sector of a Sphere, being equal to a Cone whoſe Baſe is equal to the ſpherical Surface, and its Altitude to the Radius; (Cor. 1.) it is manifeſt, that a Segment of the Sphere having the ſame portion of the Spheres Surface, as a Sector, is equal to the Difference between the Sector and a Right Cone on the Baſe of the Segment, and, its Side equal to the Radius of the Sphere.

Thus, having ſhewn how the ſolid Contents of regular geometrical Solids are obtained, the application of it, in complex ones, will not be difficult; being well verſed in the Theory, and having a ſolid foundation, for the whole, in Geometry.

7. In reſpect of Guaging; are not almoſt all large Veſſels, either Cylinders or Fruſtrums of Cones? and conſequently they are meaſured on that Principle; allowing ſo many Gallons, Quarts, &c. to a cubical Foot, as it is eaſily known to contain.

Barrels, of all kinds, are meaſured as two Fruſtrums, whoſe common Baſe is the Diameter at the Bung; but being almoſt wholly curved, from the Head to the middle, it will be truer to conſider it as four, or ſix Fruſtrums, two and two, of which, are equal; or the middle part as a Cylinder.

 8. The

8. The true ſolid Contents of irregular Solids cannot be obtained, otherwiſe than by covering them with Water, or other Fluid, in ſome regular Veſſel; and by taking it out, the Difference, in the fulneſs of the Veſſel, is its Contents.

In meaſuring the Trunks of Trees, &c. which are, generally conical, it is uſual to take the Girt in the middle, in order to reduce them to right angled Parallelopipeds; which, if the intention be to produce the real ſolid Contents, is very erroneus. It is well known, that a Circle contains the greateſt Area of any other plane Figure, having an equal Perimeter; and in Art. 10. is ſhewn the great difference between a Circle and a Square, of equal circuit. Conſequently, a Cylinder would be to a right angled Parallelopiped, of equal length, and two of its oppoſite Faces, (i. e. its Ends) Squares, whoſe Sides are, each, a fourth part of the Circumference of the Baſe, of the Cylinder, as the Circle is to the Square; the difference of which is very conſiderable. (See Art. 10. of Superficies.)

And alſo (being the Fruſtrum of a Cone) if the Girt be taken in the middle, in order to reduce it to a Cylinder, it is alſo erroneous; ſeeing, it is obvious, that the Fruſtrum of the Cone is more than ſuch a Cylinder; the exceſs of the greater End being conſiderably more than the deficiency of the other. But, as I do not intend to treat at large on thoſe matters, and having gone beyond my firſt deſign, I ſhall only obſerve, that, in this Caſe, the Fruſtrum of a Cone is equal to the Difference, between the whole Cone and the leſſer Cone, ſuppoſed to be cut off, from the Vertex.

FINIS, of MENSURATION.

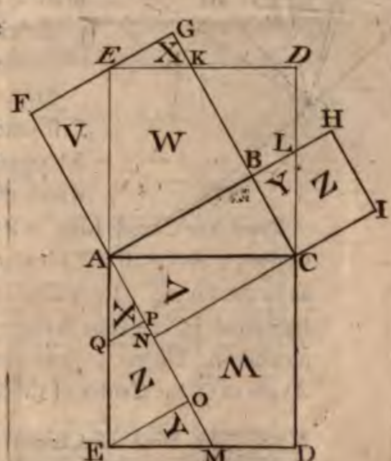
A mechanical and ocular Demonſtration of the 20th Theorem, Book 1ſt.

In Theo. 20. of the firſt Book of Elements, it is demonſtrated, that the Square of the Hypothenuſe, of every right-angled Triangle, is equal to both the Squares of the two Sides, containing the Right Angle. It may be entertaining to ſome, and conviction to others, to give ocular Demonſtration of it; by cuting the two Squares of the Sides, in ſuch wiſe, that they ſhall exactly cover the Square conſtructed on the Hypothenuſe.

On the right-angled Triangle A B C, deſcribe the three Squares, A C D E, A F G B, and B H I C.

Let the Square A E D C, of the Hypothenuſe, be ſuppoſed to be turned over, on AC, in the inverted poſition A *E* *D* C, cuting the two other Squares; B F, in A*E*, *E* K, and, the Square B I, in C L.

The two Trapeziums, W and Z, and the three Triangles V, X, and Y, will exactly coincide with the ſame Figures, as they are deſcribed on the Square A C D E.



Or, having drawn the Squares as directed, produce E A to *E*, and D C to L; and draw *E*K, perpendicular to A E. Alſo, produce FA to M, and I C to N; draw E O perpendicular to AM, make O P equal B C, (equal EC) and draw P Q parallel to EO.

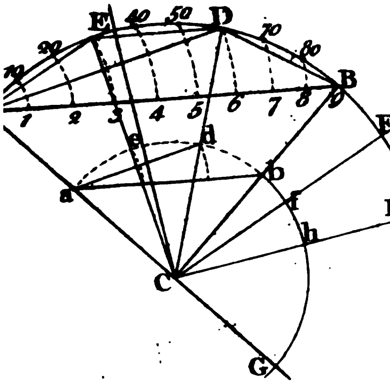
The Lines being drawn on the three Squares, as they are deſcribed, in the Figure annexed, on ſtrong Paper or Paſteboard, and the two Squares, F B and B I, being cut, carefully, with a fine thin edged knife, or otherwiſe, into five pieces, as in the Figure, they will, each, coincide with the correſponding Figures in the Square A C D E,

If the poſition of the Letters, V, W, X, &c. be regarded, there needs no other directions in reſpect of placing them.

The

The Line of Chords, and its Ufe, explained.

As the Line of Chords, and its Ufe, is better known than underftood, I prefume, a Defcription of it may not be unneceffary. And firft, I will fhew how to conftruct a Line of Chords.



With any Radius, at Difcretion, as AC, on C defcribe the Ark A D B, of 90 Degrees; i. e. make ACB a Right Angle; and on the Center, C, with any Radius, AC, draw ADB; which will be a fourth part of the circumference of a Circle.

Divide the Ark ADB into 9 equal Parts. Firft, make AD equal AC and divide DB into three equal parts, and AD will be equal fix of thofe divifions, each of which will be 10 Degrees of that Circle.

Draw the Chord Line AB; and, from the extreme A, as a Center, transfer all the Meafures on the Ark, to the Chord AB, as in the Figure. Each Divifion being fubdivided, into ten, and transfered in the fame manner, AB will then be a Scale for any number of Degrees, from one to ninety; which gives a Right Angle at C, the Center of the Circle, of which, AC is the Radius.

N. B. This Line of Chords is upon every common Scale, in a Cafe of Inftruments, for drawing; the Ufe, of which, is the fame as the Protractor, defcribed in the Theory of Plane Angles; viz. to meafure any Plane Angle; or, to lie down any Angle required, on a Plane, the meafure of the Angle, that is, the number of Degrees it contains, being known.

In applying the Line of Chords to ufe, take 60 Deg. from A, on the Line AB, with your Compaffes; and, with that Radius, on any Point, C, of the Line AC (at which Point an Angle is required, with that Line, of fome known meafure) defcribe the Ark AEB, cuting AC in A; from which Point, fet off the number of Degrees, on the Ark, which is the Meafure of the Angle required.

e. g. Suppofe an Angle of 35 Degrees is wanted, take 35 Deg. from the Point A on the Line of Chords, and fet it off from A to E, draw EC; then, ACE is an Angle of 35 Deg.

If an Angle of 60 Degrees was required; make AD equal to the Radius, and draw DC; then is ACD an Angle of 60 Degrees. For ADC is an Equilateral Triangle, whose Angles are all equal, and consequently, the Chord, AD, is equal to the Radius AC-11.4.

If a Right Angle be required; from A make AB, equal to the whole Line of Chords of 90 Degrees, and draw BC, which will be perpendicular to AC.

Thus, may any Angle, less than 90 Degrees be readily obtained. But, if an obtuse Angle is wanted, as 105 Degrees, with the Line AC, at the Point C; having first set of 90 Degrees, from A to B, or 60, to D; take the remainder BF, 15 Degrees or DF 45, and draw FC, making the Angle ACF, 105 Degrees. Thus may any Angle whatever be described.

Or, an obtuse Angle may be thus taken, at once, if there be room on your Paper.

If an Angle ACF, of 105, or ACH, 125, be required; subtract the numberof Degrees from 180, the Sum of two Right Angles, the Difference is 75, or 55.

Produce AC towards G; with the Radius AC, equal AD, (i. e. to 60 Degrees, on the Line of Chords,) describe the Ark GHF, and set off GF, 75, or GH 55 Degrees, and draw CF, or CH, making the Angle required.

After the same manner an Angle may be measured; by describing an Ark, on its Vertex, with the Radius of 60 Degrees, cuting both Sides; as AE, cuting AC and CE, of the Angle ACE; then, the measure of the Chord Line, AE (applied to AB) shews the number of Degrees that Angle contains.

The Reason of all which is so very obvious, it needs no further explanation; but, be sure that you always take your measure, or number of Degrees on the Line of Chords, begining at the Point A; because it is evident, that they are continually diminishing, from A to B, although they are equal on the Ark; for the Divisions on AB are the Chords of each Ark, set off from A towards B; each of which, is still less in proportion to the Ark.

The Chord of 10 Degrees, it is evident, deviates very little from a Right Line; the Chord of 30 Degrees deviates confiderably, and AD, the Chord of 60, ftill more; for the Chord AD, of 60, and the two Chords of 30 make a Triangle, AED; of which, it is manifeft, that, the two Sides AE, ED, each of 30, Degrees is greater than AD, the Chord of 90. (13. 1. El.)

If it fhould be afked, why 60 Degrees, particularly, is taken for Radius, when we begin the Operation of taking or defcribing an Angle, by the Line of Chords? the Reafon is plain; becaufe the Chord of 60 Degrees is always equal to the Radius; and, the Side, AD, of the Equilateral Triangle ADC, is the Side of a regular Hexagon infcribed in a Circle, of that Radius; and confequently, it is equal to a fixth Part of the Circumference, or 60 Degrees. Hence it is clear, that no other Radius can anfwer to that Line of Chords.

FINIS.

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At the Author's Houfe, in Poland-Street, Oxford-Road, the third from Broad-Street on the Left, Youth are taught Geometry and Menfuration; Trigonometry, plane and fpherical; Projection of the Sphere, &c. Architecture and Perfpective; alfo the Ufe of the Globes, and Explanation of the Orrery, &c.

Ladies and Gentlemen taught at their own Home, or as above, in the moft expeditious and familiar manner.

He will make any Gentleman (having a Talent for it) a Proficient in Geometry, in lefs than half the Time ufually fpent in it; making it, at the fame time, inftructive and entertaining.

Drawings, in Perfpective; Architectural; or of Models, Machines, Mathematical Inftruments, &c. executed with the greateft accuracy.

Perfpective Views of Gentlemens Seats, &c. by himfelf and Son.